BAND WIDTH ESTIMATES

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Abstract. As is well known, a positive Ricci curvature bound implies a diameter bound for a complete manifold. When the Ricci curvature is replaced by the scalar curvature, such a diameter bound is not possible. However, a bound on the size of the manifold along a certain direction is possible if we also impose a special topology. This is known as Gromov's band width estimate. There are various proofs of the band width estimate, and we focus the prescribed mean curvature surface (or μ-bubble) approach to the band width estimate of Gromov. The first lecture will introduce some basics of hypersurfaces in a Riemannian manifold, in particular, first variation and second variation of μ-bubbles, torical scalar curvature rigidity and foliation construction. The key results of the first lecture were due to Schoen-Yau, Bray-Brendle-Neves, Gromov and J. Zhu.

We introduce the weighted μ-bubble and apply the technique to study band width estimate under a spectral condition. If time permits, we will also discuss some Llarull type theorems, and spacetime settings of the band width estimates.

1. Basics of hypersurfaces in Riemannian manifold

We assume familiarity with concepts in Riemannian manifold such as metrics, Levi-Civita connections. See for example [dC92], [Pet98]. Now we review basics of hypersurfaces in a Riemannian manifold.

1.1. Quick review of Riemannian geometry.

Let g be a Riemannian metric on the manifold M and ∇ be the related Levi-Civita connection. We define the Riemann curvature ${R_{ijk}}^l$,

$$ {R_{ijk}}^l \partial_l := \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k,\ R_{ijkl} = g_{lq}{R_{ijk}}^q. $$

The Ricci curvature is

$$ \mathrm{Ric}_{il} = g^{jk} R_{ijkl}. $$

The scalar curvature is

$$ R = g^{il}\,\mathrm{Ric}_{il}\,. $$

1.2. **Hypersurface.** Let Σ be a hypersurface in (M, g), and ν be a unit normal to Σ, then the **second fundamental form** of Σ in M is given by

$$ A(X,Y) = -\langle \nabla_X Y, \nu\rangle. $$

Then the **mean curvature** is

$$ H = -\sum_{i=1}^{n-1} \langle \nabla_{e_i} e_i, \nu\rangle, $$

where $\{e_i\}$ is an orthonormal frame of the tangent space of Σ. We can choose an orthonormal frame such that $A(e_i, e_j) := A_{ij}$ is diagonalized, the diagonal entry is called **principle curvatures**.

1.3. **First variation of area.** Let Σ be a hypersurface in M, let Σ_t be a one-parameter family of deformations of Σ given by $\phi(\Sigma, t)$ with $\phi(\Sigma, 0) = \Sigma$. Let $\{x_1, \dots, x_n\}$ be a coordinate system around a point $p \in \Sigma$. We can consider $\{x_1, \dots, x_n, t\}$ to be a coordinate system of $\Sigma \times (-\varepsilon, \varepsilon)$ near the point $(p, 0)$. Let

$$e_i = \mathrm{d}\phi(\tfrac{\partial}{\partial x^i}),\ T = \mathrm{d}\phi(\tfrac{\partial}{\partial t}).$$

The induced metric on Σ is given by

$$\sigma_{ij} = \langle e_i, e_j \rangle.$$

We know that

$$\begin{aligned}\partial_t \sigma_{ij} =& T\langle e_i, e_j\rangle \\ =& \langle \nabla_T e_i, e_j\rangle + \langle e_i, \nabla_T e_j\rangle \\ =& \langle \nabla_{e_i} T, e_j\rangle + \langle e_i, \nabla_{e_j} T\rangle.\end{aligned}$$

The variation of the area element is given by

$$\partial_t \sqrt{\det \sigma} = \sigma^{ij} \langle \nabla_{e_i} T, e_j\rangle \sqrt{\det \sigma},$$

where σ^{ij} is the inverse of σ_{ij}. The vector T can be decomposed to two components which are respectively tangent to Σ_t and normal to Σ_t. When T is normal to Σ, we call Σ_t a **normal variation**. We see that

$$\partial_t \sqrt{\sigma} = \mathrm{div}_\Sigma T^\top + \mathrm{div}_\Sigma T^\perp = \mathrm{div}_\Sigma T^\top + \langle T, \nu\rangle H, \tag{1.1}$$

where $\mathrm{div}_\Sigma = \sigma^{ij} \langle \nabla_{e_i}(\cdot), e_j\rangle$.

1.4. **Gauss equation.** The Gauss equation

$$\bar{R}_{ijkl} = R_{ijkl} - h_{jk} h_{il} + h_{ik} h_{jl}.$$

Lemma 1.1 (Schoen-Yau rewrite)**.**

$$\mathrm{Ric}_M(\nu, \nu) = \frac{1}{2} R_M - \frac{1}{2} R_\Sigma - \frac{1}{2} |A|^2 + \frac{1}{2} H^2. \tag{1.2}$$

Proof We calculate directly by using definitions and Gauss equation. We again use an orthonormal frame. We have

$$\begin{aligned}& \mathrm{Ric}_M(\nu, \nu) \\ =& \sum_i \mathrm{Rm}_{i\nu\nu i} \\ =& \sum_{i,j} \mathrm{Rm}_{ijji} - \sum_{j \neq \nu} \mathrm{Rm}_{ijji} \\ =& R_M - \sum_{j\neq\nu, i\neq\nu} \mathrm{Rm}_{ijji} - \sum_{j\neq\nu} \mathrm{Rm}_{\nu jj\nu} \\ =& R_M - \sum_{j\neq\nu, i\neq\nu} \mathrm{Rm}_{ijji} - \sum_{j} \mathrm{Rm}_{\nu jj\nu} \\ =& R_M - \sum_{j\neq\nu, i\neq\nu} \mathrm{Rm}_{ijji} - \mathrm{Ric}_M(\nu, \nu).\end{aligned}$$

Hence,

$$2\operatorname{Ric}_M(\nu,\nu) = R_M - \sum_{j\neq\nu, i\neq\nu} \mathrm{Rm}_{ijji}$$

Using Gauss equation on the second term, we see

$$\mathrm{Rm}_{ijji} = \mathrm{Rm}^{\Sigma}_{ijji} + h_{ij}h_{ij} - h_{ii}h_{jj}.$$

Doing summation and combining the two equations above finishes the proof. $\square$

Lemma 1.2. *Let A be any* ***symmetric*** *2-tensor on Σ^{n-1}, let $\operatorname{tr} A = \sigma^{ij}A_{ij}$ be the trace of A, then*

$$\sigma^{ik}\sigma^{jl}A_{ij}A_{kl} = |A|^2 \geqslant \tfrac{1}{n-1}(\operatorname{tr} A)^2. \tag{1.3}$$

Proof First, we need to show that both sides of the inequality (1.3) does not depend on the choice of frames. (**Exercise**)

We can choose a frame such that σ_{ij} is the identity matrix (i.e. the frame $\{e_i\}$ is an orthonormal frame) and $A_{ij} = \kappa_i\sigma_{ij}$, then

$$|A|^2 = \sum_i \kappa_i^2, (\operatorname{tr} A)^2 = (\sum_i \kappa_i)^2.$$

Then (1.3) follows from elementary arithmetic-geometric mean inequality. $\square$

Exercise 1.1. *Let $A^0 = A - \frac{1}{n-1}(\operatorname{tr} A)\sigma$ (A^0 is called* ***traceless part*** *of A), prove that $|A|^2 = \frac{1}{n-1}(\operatorname{tr} A)^2 + |A^0|^2$.*

1.5. **Second variation of area of minimal hypersurfaces.** Essentially, we are computing the first variation of the mean curvature according to (1.1).

Lemma 1.3. *We have that*

$$X(H) = -\Delta f - (\operatorname{Ric}_M(\nu,\nu) + |A|^2)f, \tag{1.4}$$

assuming that $X = f\nu$ along Σ.

Proof We calculate directly, assume that $\{e_i\}$ is an orthonormal frame along Σ, then

$$\begin{aligned}
&\nabla_X(\sigma^{ij}\langle\nabla_{e_i}\nu, e_j\rangle)\\
=&-X(\sigma^{ij})h_{ij} + \sigma^{ij}\nabla_X\langle\nabla_{e_i}\nu, e_j\rangle\\
=&-2h_{ij}h_{ij} + \sigma^{ij}\langle\nabla_X\nabla_{e_i}\nu, e_j\rangle + \sigma^{ij}\langle\nabla_i\nu, \nabla_X e_j\rangle\\
=&-2|A|^2 + \sigma^{ij}\operatorname{Rm}(X, e_i, \nu, e_j) + \sigma^{ij}\langle\nabla_i\nabla_X\nu, e_j\rangle + \sigma^{ij}\langle\nabla_i\nu, \nabla_j(\phi\nu)\rangle.
\end{aligned}$$

It finishes the proof if we can prove $\nabla_X\nu = -\nabla\phi$. Indeed,

$$\langle\nabla_X\nu, e_i\rangle = -\langle\nabla_X e_i, \nu\rangle = -\langle\nabla_i X, \nu\rangle = -\nabla_i\phi.$$

$\square$

We say that a (closed) minimal surface is **stable** if $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\operatorname{vol}(\Sigma_t)|_{t=0} \geqslant 0$, i.e.

$$\int_\Sigma |\nabla f|^2 - (\operatorname{Ric}_M(\nu) + |A|^2)f^2 \geqslant 0, \tag{1.5}$$

for any $f \in C^\infty(\Sigma)$. We have (see [CM11])

Lemma 1.4. *The following three are equivalent:*

1) The stability (1.5) *holds;*

2) The operator $L := -\Delta - (\mathrm{Ric}_M(\nu) + |A|^2)$ *has a non-negative first eigenvalue (principal eigenvalue);*

3) There exists a positive C^2 *function* ϕ *such that* $L\phi \geqslant 0$.

If a minimal surface is stable, the third item says that we can increase the mean curvature if deforming the hypersurface Σ in the direction $\phi\nu$.

1.6. **Warped product.** We say that $(M = [t_-, t_+] \times \Sigma, \bar{g} = \mathrm{d}t^2 + \phi(t)^2 g_0)$ is a **warped product**. Here g_0 is a metric on Σ.

Lemma 1.5. *The scalar curvature of* $(M^n, \bar{g})$ *is given by (assuming that* Σ *is of dimension* n*) is*

$$R_g = R_\Sigma \phi^{-2} - n(n-1)(\phi'/\phi)^2 - 2(n-1)(\phi'/\phi)'. \tag{1.6}$$

Proof We can calculate directly. Here, we use (1.4). Let $\Sigma_t = \{t\} \times \Sigma$, then the induced metric of $\bar{g}$ is

$$\sigma = \phi(t)^2 g_0.$$

So the second fundamental form is

$$h = \tfrac{1}{2}\partial_t \sigma = \tfrac{\phi'}{\phi} g_0,$$

and the mean curvature of Σ_t is

$$H = \mathrm{tr}(\sigma^{-1}h) = (n-1)\tfrac{\phi'}{\phi}.$$

So the scalar curvature of $(M, \bar{g})$ by (1.4) is

$$R_g = R_\Sigma \phi^{-2} - H^2 - |A|^2 - 2\partial_t H = R_\Sigma \phi^{-2} - n(n-1)(\phi'/\phi)^2 - 2(n-1)(\phi'/\phi)',$$

where we also have used (1.2). □

Question 1.1 ([Gro18], Gromov's rigid band for his band width estimate)**.** *Find the function* ϕ *such that the scalar curvature of the metric* $\mathrm{d}t^2 + \phi(t)^2 g_{\mathbb{T}^{n-1}}$ *is* $n(n-1)$*. How about* $-n(n-1)$ *and* 0*?*

Question 1.2. *Find the scalar curvature of* $\phi(t)^2(\mathrm{d}t^2 + g_0)$ *using* (1.4) *and* (1.2).

2. Three dimensional Geroch conjecture

2.1. Gauss-Bonnet theorem.

Theorem 2.1. *Let* Σ *be a closed oriented surface, then*

$$2\pi\chi(\Sigma) = \int_\Sigma K$$

where $K = \frac{1}{2}R_\Sigma$ *is the Gauss curvature.*

Corollary 2.2. *Let* Σ *be a 2-torus with* $\mathrm{Sc}_g \geqslant 0$*, then* Σ *must be flat.*

2.2. **Geroch conjecture.** Below is a generalization of the two dimensional rigidity (Corollary 2.2) made by Geroch in 1975.

Conjecture 2.3. *On $\mathbb{T}^n$, there does not exist a metric g with $R_g \geqslant 0$ except the flat ones.*

The conjecture was confirmed by Schoen-Yau [SY79] and Gromov-Lawson [GL83] via different methods. See also Stern [Ste22] for a proof by harmonic 1-forms in three dimensions.

2.3. **Proof of Geroch conjecture in dimension 3.** We show a proof of Geroch conjecture by introducing some modern ideas [SY79], [FS80], [BBN10].

Theorem 2.4. *On $\mathbb{T}^3$, there does not exist a metric g with $R_g \geqslant 0$ except the flat ones.*

Proof Let $(x_1, \ldots, x_3)$ be the parameters of the torus, then $\mathrm{d}x^1$ is dual to an element in $H_2(M;\mathbb{Z})$ (taking cap product with the fundamental class). We minimize area in $H_2(M;\mathbb{Z})$ and we obtain an area-minimizing surface Σ. By the stability inequality,

$$Q(f,f) := \int_\Sigma |\nabla f|^2 - (\mathrm{Ric}_M(\nu) + |A|^2) f^2 \geqslant 0. \tag{2.1}$$

By taking $f \equiv 1$ and using the (1.2), we see that

$$\tfrac{1}{2}\int_\Sigma R_\Sigma \geqslant \tfrac{1}{2}\int_\Sigma (R_g + |A|^2) \geqslant 0.$$

Using the Gauss-Bonnet theorem on the left, we see that

$$2\pi\chi(\Sigma) = \tfrac{1}{2}\int_\Sigma R_\Sigma \geqslant \tfrac{1}{2}\int_\Sigma (R_g + |A|^2) \geqslant 0.$$

But the left is less than or equal to zero, so $R_g = |A|^2 = 0$ along Σ. Putting these information back to (2.1), we see that

$$Q(1,1) = 0.$$

This implies that $f = 1$ is the eigenfunction and zero is the lowest eigenvalue of

$$L := -\Delta - (\mathrm{Ric}_M(\nu) + |A|^2) = -\Delta + \tfrac{1}{2} R_\Sigma.$$

This implies that $R_\Sigma = 0$ as well.

A **remark**: if L (linearization of H) is an invertible operator, by inverse function theorem, then we can deform Σ such that the mean curvature of the deformed surface attain any value near H_Σ. **However**, L is not invertible and constant functions are the kernel of L.

Let Z be a vector field near Σ such that $Z = \nu$ along Σ and $\psi = \psi(x,t)$ be the flow of Z.

Let $\Sigma_u = \{\phi(x, u(x))\}$ which is properly embedded if u has small norm. Denote every quantity related to Σ_u by the subscript u.

Let $E = \{u \in C^{2,\alpha} : \int_\Sigma u = \langle u, 1\rangle_{L^2} = 0\}$ and $F = \{u \in C^{0,\alpha} : \int_\Sigma u = 0\}$. We define

$$\Phi : (-\varepsilon, \varepsilon) \times E \to F$$

$$(t,u) \mapsto H_{t+u} - \tfrac{1}{|\Sigma|}\int_\Sigma H_{t+u}.$$

We calculate $D\Phi|_{(0,0)}$. For $u \in E$, define $f : (s,x) \in (-\varepsilon, \varepsilon) \times \Sigma \mapsto \phi(x, sv(x))$ gives a variation of Σ and the vector field

$$\tfrac{\partial f}{\partial s}|_{s=0} = uZ = u\nu \text{ on } \Sigma.$$

We see then

$$D\Phi|_{(0,0)}(0,u) = \tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\Phi(0,su) = -\Delta u + \tfrac{1}{|\Sigma|}\langle \Delta u, 1\rangle_{L^2} = -\Delta u.$$

This is an isomorphism restricted to $0 \times E$. Implicit function theorem implies that for small t, there exists $u(t)$ with small norm such that $\Phi(t, u(t)) = 0$, $u(0) = 0$, $\Phi(0,0) = 0$. In other words, $\Sigma_{t+u(t)}$ is constant mean curvature for each t.

Let $w : (t,x) \mapsto t + u(t)(x)$, since

$$H_{w(\cdot,t)} - \tfrac{1}{|\Sigma|}\int_\Sigma H_{w(\cdot,t)} = \Phi(t, u(t)) = 0.$$

By taking derivative with respect to t, we see that $\frac{\partial w}{\partial t}|_{t=0}$ satisfies $\Delta(\frac{\partial w}{\partial t}|_{t=0}) = 0$, therefore, must be a constant.. Since

$$\int_\Sigma (w(x,t) - t) = \int_\Sigma u = 0.$$

So $\frac{\partial w}{\partial t}|_{t=0} = 1$. So $\Sigma_{t+u(t)}$ is a foliation. Now set $\lambda(t) = H_{t+u(t)}$, $\Sigma_t = \Sigma_{t+u(t)}$ then

$$\lambda'(t) v_t^{-1} = -v_t^{-1}\Delta_{\Sigma_t} v_t - (\mathrm{Ric}(\nu_t) + |A_{\Sigma_t}|^2).$$

We first use the rewrite of Schoen-Yau, it leads to

$$\lambda'(t)\tfrac{1}{v_t} = -v_t^{-1}\Delta_{\Sigma_t} v_t - \tfrac{1}{2}(R_M - R_{\Sigma_t} + |A_{\Sigma_t}|^2 + H_{\Sigma_t}^2),$$

with an integration, integration by parts,

$$\lambda'(t)\int_{\Sigma_t}\tfrac{1}{v_t} \leqslant -\int_{\Sigma_t} v_t^{-1}\Delta_{\Sigma_t} v_t + \int_{\Sigma_t} R_{\Sigma_t} = -\int_{\Sigma_t}\frac{|\nabla_{\Sigma_t} v_t|^2}{v_t^2} + 4\pi\chi(\Sigma_t) \leqslant 0.$$

So $\lambda'(t) \leqslant 0$ for $t \geqslant 0$. So the mean curvature is decreasing and $H_{\Sigma_t} = \lambda(t) \leqslant 0$.

Now we return to the first variation of area (1.1),

$$\tfrac{\mathrm{d}}{\mathrm{d}t}|\Sigma_t| = \int_{\Sigma_t} H_{\Sigma_t} v_t \leqslant 0.$$

But $\Sigma = \Sigma_0$ is a minimiser, so every Σ_t is a minimiser. It is not difficult to see that the foliation is equi-distant and extends to the whole manifold. $\square$

Remark 2.5. The proof works through if we replace $\mathbb{T}^3$ with the connected sum $\mathbb{T}^3 \sharp M$ where M is a closed 3-manifold.

Remark 2.6. People from PDE background may recognize the construction of constant mean curvature foliation is a geometric version of the Lyapunov-Schmidt reduction method.

Corollary 2.7. *There does not exist a metric on $\mathbb{T}^3 \sharp M$ such that g has strictly positive scalar curvature.*

Corollary 2.8. *Assume that the Geroch conjecture holds for n dimensional manifold $\mathbb{T}^n \sharp M$, there does not exist a metric on $\mathbb{T}^n \sharp M$ such that if the first eigenvalue of the conformal Laplacian*

$$-\Delta + c_n R_g, c_n = \tfrac{n-2}{4(n-1)}$$

is positive.

Proof Let $u > 0$ be the first eigenfunction of $-\Delta + c_n R_g$, then

$$\mathrm{Sc}(u^{\frac{4}{n-2}} g) = c_n^{-1} u^{-\frac{n+2}{n-2}}(-\Delta u + c_n R_g u) > 0. \tag{2.2}$$

The metric $u^{\frac{4}{n-2}} g$ contradicts the previous corollary. □

Exercise 2.1. *This is a classic exercise in Riemannian geometry: prove the relation* (2.2).

Theorem 2.9. *On $\mathbb{T}^n$ ($3 \leqslant n \leqslant 7$), there does not exist a metric g with $R_g \geqslant 0$ except the flat ones.*

The proof would not be given. The theorem is also valid for a manifold which admits a map of non-zero degree to $\mathbb{T}^n$.

3. Basic theory of μ-bubbles

Let Ω' be an open set with smooth $\partial\Omega'$, we consider the functional

$$E(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) - \int (\chi_\Omega - \chi_{\Omega'}) h \mathrm{d}\mathcal{H}^n \tag{3.1}$$

for all sets with finite perimeter such that $\Omega \triangle \Omega'$ is compactly supported away from the boundary ∂M. Here h is a Lipschitz function on M. For a reference on sets of finite perimeter, see [Giu84]. Without loss of generality, we can assume Ω is smooth.

Let Ω_t be a smooth family of sets, such that $\Omega' \subset \Omega_t$ for all t and $\Omega_0 = \Omega$, the first variation is given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E(\Omega_t)|_{t=0} = \int_{\partial\Omega} (H - h)\phi \mathrm{d}\mathcal{H}^{n-1}. \tag{3.2}$$

We say that Ω is a $h-$**bubble** if the first variation vanishes. Sometimes, without explicit references to the function h, we call Ω a μ**-bubble**. We call Σ a **surface of prescribed mean curvature** h.

We call a μ-bubble is **stable** if the **second variation** $\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t)|_{t=0} \geqslant 0$. We calculate the second variation explicitly. Assume that variational vector field is X, then the integrand of the second variation is given by

$$X((H - h)\phi \mathrm{d}\mathcal{H}^{n-1}) = X(H - h)\phi \mathrm{d}\mathcal{H}^{n-1},$$

since $H = h$ along $\partial\Omega$. Using (1.4) and $X(h) = \phi \nabla_\nu h$, we see

$$\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t)|_{t=0} = \int_{\partial\Omega} (-\Delta\phi - \mathrm{Ric}_M(\nu)\phi - |A|^2\phi - \nabla_\nu h)\phi. \tag{3.3}$$

4. Gromov band width estimate

4.1. **Band.** We introduce a geometric object initiated by Gromov [Gro18] called a **band**.

Definition 4.1. *A band is an orientable, compact manifold M such that its boundary ∂M consists at least two connected components. Let $\partial_- M$ be a union of some connected components of $\partial_- M$, and $\partial_+ M = \partial M \sim \partial_- M \neq \emptyset$. Then band width is given by*

$$\mathrm{wi\,d\,th}(M, g) = \mathrm{dist}(\partial_- M, \partial_+ M).$$

Note that the band width depends on the choice of $\partial_- M$ and $\partial_+ M$. See the following for an example of bands.

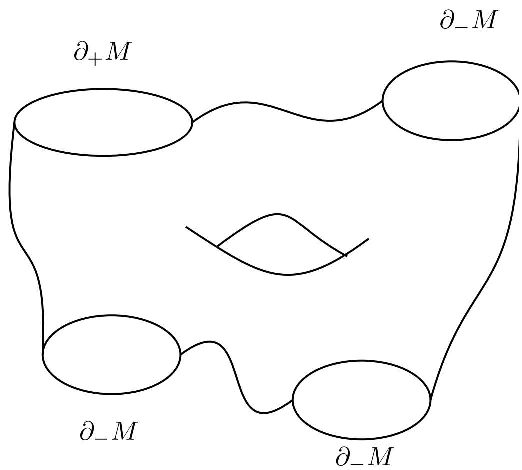

FIGURE 1. A band with four boundary components.

Let $M_0 = \mathbb{T}^{n-1} \times [-1, 1]$ and $\partial_\pm M_0 = \mathbb{T}^{n-1} \times \{\pm 1\}$. We just call M_0 a **torical band**.

Gromov also introduced the **over-torical band**: If M admits a continuous map of non-zero degree $f : (M, \partial_\pm M) \to (M_0, \partial_\pm M_0)$, then M is called an over-torical band.

4.2. **Existence and regularity theory of μ-bubbles.**

We have the following standard result from geometric measure theory.

Theorem 4.2. *For a manifold M with at least two boundary components, we assume that $M = \partial_- M \cup \partial_+ M$. If $H_{\partial_- M} < h$ along $\partial_- M$ ($H_{\partial_- M}$ is calculated with respect to the inward unit normal), $H_{\partial_+ M} > h$ along $\partial_+ M$, then there exists a minimiser Ω to the action* (3.1) *and $\partial\Omega$ lies away from ∂M. When the dimension $3 \leqslant n \leqslant 7$, $\partial\Omega$ is free of singularities and $\partial\Omega$ is a hypersurface with regularity $C^{2,\alpha} \cap W^{3,p}$. Moreover, $\partial\Omega$ is homologous to $\partial_\pm M$.*

Now there are two choices of unit normals to compute the mean curvature H of $\partial\Omega$, in the above theorem, it is chosen so that it points to the inside the region bounded by $\partial\Omega$ and $\partial_+ M$. Note that $-(H_{\partial_- M} - h)\nu_{\partial_- M}$ points inside of M, we can run the mean curvature flow F_t,

$$\partial_t F = -(H - h)\nu,$$

which starts from $\partial_- M$. We know the short time existences of the flow. If $H_{\partial_- M} - h \leqslant 0$, we can immediately have along $\partial\Omega$ that $H_{F_t} < h$ by the maximum principle. Hence, we can improve Theorem 4.2 a little bit.

4.3. **Gromov's Band width estimate.**

Theorem 4.3 ([Gro18]). *Let $(-1,1)\times\mathbb{T}^{n-1}$ carry a metric with $\mathrm{Sc}_g \geqslant n(n-1)$, then the band width of M is less than $\frac{2\pi}{n}$.*

Gromov proved this band width estimate for over-torical bands, for simplicity, we only state and prove for just torical bands. Our proof is based on the geometric applications of the μ-bubble technique, and essentially it boils down to a good choice of h.

One naturally wonders what if the band width $2\pi/n$ is achieved. We introduce the notion of a rigid band. A **rigid band** is a band which realizes the lower bound of the (here, scalar) curvature and the width. The rigid band associated with Theorem 4.3 is given by

$$((-\tfrac{\pi}{n},\tfrac{\pi}{n})\times\mathbb{T}^{n-1}, \mathrm{d}t^2+\cos^{\frac{4}{n}}(\tfrac{nt}{2})g_{\mathbb{T}^{n-1}}).$$

See Exercise 1.1. Let $\eta(t)$ be the mean curvature of the level set of $\mathrm{d}t^2+\cos(\frac{nt}{2})^{4/n}g_{\mathbb{T}^{n-1}}$. The very important property of η is that it satisfies the ODE

$$n(n-1)+\tfrac{n}{n-1}\eta^2+2\eta'=0 \text{ and } \eta'<0. \tag{4.1}$$

Now this ODE is essentially (1.6): $\eta=(n-1)\phi'/\phi$.

Before we go on we need to fix some orientation: We fix Ω to be the region bounded by $\partial_- M$ and a surface Σ homologous to $\partial_- M$.

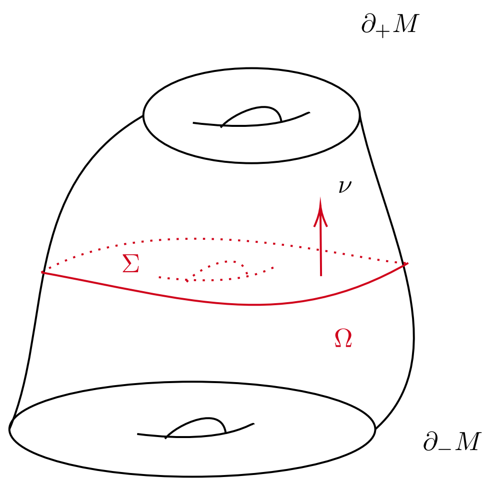

Figure 2. What is Ω?

We fix the direction of the unit normal of Σ such that it always points outside of Ω.

Proof Assume on the contrary that $\mathrm{width}(M,g)>\frac{2\pi}{n}$, then

$$d(x)=b\min\left\{\max\{\mathrm{dist}(x,\partial_- M)-\tfrac{\pi}{n}-\varepsilon,-\tfrac{\pi}{n}\},\tfrac{\pi}{n}\right\} \tag{4.2}$$

where $0<2\varepsilon<\mathrm{width}(M,g)-\frac{2\pi}{n}$, where $0<b<1$ is sufficiently close to 1.

The construction of this function seems a bit clumsy, geometrically, we just need this function to be linear with respect to the distance function from the boundary and taking values $[-\frac{\pi}{n}, \frac{\pi}{n}]$ a positive distance away from $\partial_\pm M$. We also know that $|\nabla d| < 1$.

Let $M_1 = d^{-1}([-\frac{\pi}{n}, \frac{\pi}{n}])$, and we set $\partial_\pm M_1 = d^{-1}(\pm\frac{\pi}{n})$ and $h = \eta(d(x))$. We see that $h(\partial_\pm M) = \mp\infty$. Hence, by Theorem 4.2, we can find a stable surface Σ of prescribed mean curvature h in $M_1 \subset M$. Note that Σ is stable, so the second variation (3.3) is non-negative, that is,

$$\int_\Sigma (|\nabla\phi|^2 - (\mathrm{Ric}_M(\nu,\nu) - |A|^2 + \nabla_\nu h)\phi^2) \geqslant 0.$$

By the (1.2) and (1.3),

$$\mathrm{Ric}_M(\nu,\nu) + |A|^2 = \tfrac{1}{2}(R_M - R_\Sigma + |A|^2 + H^2) \geqslant \tfrac{1}{2}(R_M - R_\Sigma + \tfrac{n}{n-1}h^2).$$

Now an estimate on $\nabla_\nu h$ is as follows,

$$-\nabla_\nu h = -\eta'(d(x))\langle\nabla d, \nu\rangle \geqslant \eta'(d(x))|\nabla d| > \eta'(d(x)), \tag{4.3}$$

because of $|\nabla d| < 1$ and $\eta' < 0$. So putting the above three equations together yields

$$\int_\Sigma (|\nabla\phi|^2 + \tfrac{1}{2}R_\Sigma\phi^2) \geqslant \tfrac{1}{2}\int_\Sigma (R_M + \tfrac{n}{n-1}\eta^2(\phi(x)) + 2\eta'(\phi(x))) > 0,$$

where we have used $R_M \geqslant n(n-1)$. We know that this is impossible because of

$$\int_\Sigma (|\nabla\phi|^2 + \tfrac{1}{2}R_\Sigma\phi^2) \leqslant 0,$$

a consequence of Corollary 2.8. Hence, the width of (M, g) is less or equal to $\frac{2\pi}{n}$.□

Question 4.1. *How to prove a version of Theorem 4.3 for over-torical bands?*

Question 4.2. *Let* $M = [t_-, t_+] \times \mathbb{T}^{n-1}$. *Assume that* M *carry a smooth metric* g *such that* $\mathrm{Sc} \geqslant n(n-1)$, *and* $H_{\partial_- M} \leqslant \eta(t_-)$ *and* $H_{\partial_+ M} \geqslant \eta(t_+)$ *where the mean curvatures of* $\partial_\pm M$ *are computed with respect to the normals pointing to the same direction as* ∂_t, η *is given by* (4.1), $t_\pm \in (-\frac{\pi}{n}, \frac{\pi}{n})$. *Then* $\mathrm{width}(M, g) \leqslant t_+ - t_-$. *Is there a rigidity statement for this band width estimate?*

(Hint: construct a similar function ϕ *as in* (4.2) *and find a surface of prescribed mean curvature* h*.)*

Question 4.3. *Formulate a similar theorem for* $\mathrm{Sc} \geqslant -n(n-1)$, $\mathrm{Sc} \geqslant 0$.

(Hint: rigid bands are $\mathrm{d}t^2 + \sinh(\frac{nt}{2})^{\frac{4}{n}} g_{\mathbb{T}^{n-1}}$ *and* $\mathrm{d}t^2 + (\frac{nt}{2})^{\frac{4}{n}} g_{\mathbb{T}^{n-1}}$*.)*

4.4. **Zhu's band width estimate.**

Theorem 4.4 ([Zhu21]). *Let* $(-1,1)\times\mathbb{T}^2$ *carry a metric with* $\mathrm{Ric}\geqslant 2$*, then the band width is less than* $\pi/2$.

Gromov [Gro18] originally conjectured this band width estimate to be valid for a sectional curvature bound $\sec\geqslant 1$. It is interesting that this rigid band in Theorem 4.4 given by

$$\mathrm{d}t^2+\sin^{1+\lambda}t\cos^{1-\lambda}t\mathrm{d}s_1^2+\sin^{1-\lambda}t\cos^{1+\lambda}t\mathrm{d}s_2^2,\ \text{where } 0\leqslant\lambda\leqslant 1,\ 0<t<\tfrac{\pi}{2}, \tag{4.4}$$

which are a family of metrics. The mean curvature of each t-level set is

$$\eta(t)=\cot t-\tan t.$$

Moreover, $\eta'<0$ and $\eta'+\eta+4=0$.

Proof We argue again by contradiction and assume that the width is larger than $\pi/2$, and as in Theorem 4.3, we construct a function d which takes values $[0,\pi/2]$ a positive distance away from $\partial_\pm M$ and $|\nabla d|<1$. Take $h=\eta(d(x))$.

Now find a surface Σ of prescribed mean curvature h. But now we use the following

$$\mathrm{Ric}(\nu,\nu)+|A|^2=\mathrm{Ric}(e_1,e_2)+\mathrm{Ric}(e_2,e_2)+H^2-R_\Sigma,$$

where $\{e_1,e_2\}$ is an arbitrary orthonormal frame of Σ. Now, $\mathrm{Ric}(e_1,e_1)\geqslant 2$ and $\mathrm{Ric}(e_2,e_2)\geqslant 2$ and $H=h(\phi(x))$. As (4.3),

$$-\nabla_\nu h>\eta'(\phi(x)).$$

The rest is straightforward. □

Question 4.4. *Compute the Ricci curvatures of* (4.4)*. What is the metric if* $\lambda=0$*, 1?*

Question 4.5. *Formulate a band width estimate with* $\mathrm{Ric}\geqslant -2$*,* $\mathrm{Ric}\geqslant 0$ *as in Exercise 4.3. (unpublished, due to Yukai Sun and myself)*

5. Rigidity analysis of Gromov band width estimate

Recall Gromov's band width estimate:

Theorem 5.1 ([Gro18]). *Let* $(-1,1)\times\mathbb{T}^{n-1}$ *carry a metric with* $\mathrm{Sc}_g\geqslant n(n-1)$*, then the band width of* M *is less than* $\frac{2\pi}{n}$.

Recall that the **rigid band** is

$$\mathrm{d}t^2+\cos(\tfrac{nt}{2})^{\frac{4}{n}}g_{\mathbb{T}^{n-1}},t\in(-\tfrac{\pi}{n},\tfrac{\pi}{n}).$$

The level set has mean curvature

$$\eta(t)=-(n-1)\tan(\tfrac{nt}{2}).$$

We note that $\eta(\pm\frac{\pi}{n})=\mp\infty$, $\eta'<0$ and it satisfies the ODE

$$n(n-1)+\tfrac{n}{n-1}\eta^2+2\eta'=0.$$

We restrict our discussion of rigidity to dimension three.

5.1. **Assume the existence of a minimiser.**

Set $d(x)=\min\{\frac{\pi}{n},-\frac{\pi}{n}+\mathrm{dist}(x,\partial_- M)\}$ and $h=\eta(d(x))$. We assume that there exists a minimiser Ω_0 to the functional

$$E(\Omega)=\mathrm{Area}(\partial\Omega)-\int_\Omega h.$$

Then by the first variation, we have that $\Sigma=\partial\Omega_0$ satisfies

$$H-h=0 \text{ along } \Sigma.$$

The second variation gives

$$\int_\Sigma [|\nabla\phi|^2-(\mathrm{Ric}(\nu)+|A|^2+\nabla_\nu h)\phi^2]\geqslant 0.$$

Using Schoen-Yau rewrite and

$$\nabla_\nu h=\eta'(d(x))\nabla_\nu d\geqslant \eta'(d(x)),$$

we see that

$$\begin{aligned}&\int_\Sigma[|\nabla\phi|^2+\tfrac{1}{2}R_\Sigma\phi^2]\\ \geqslant&\frac{1}{2}\int_\Sigma(R_g+|A|^2+H^2+2\nabla_\nu h)\\ \geqslant&\frac{1}{2}\int_\Sigma(R_g+\tfrac{3}{2}\eta(d(x))^2+2\eta'(d(x)))\geqslant 0\end{aligned}$$

by also the ODE which is satisfied by η. By using $\phi=1$ and Gauss-Bonnet theorem, we see that all inequalities must be equalities. That includes:

$$\begin{aligned}H&=h,\\ \nabla_\nu d&=1,\\ A-\tfrac{1}{2}H&=0,\\ R_g&=6,\\ R_\Sigma&=0\end{aligned}$$

along Σ.

The linearization/first variation of $H-h$ is given by

$$\nabla_{\phi\nu}(H-h)=-\Delta_\Sigma\phi-(\mathrm{Ric}(\nu)+|A|^2)\phi-\phi\nabla_\nu h, \tag{5.1}$$

and is reduced to only the Laplacian.

This allows us to construct surfaces $\{\Sigma_t\}_{t\in[-\varepsilon,\varepsilon]}$ of constant $H-h$ nearby Σ which also forms a foliation. The foliation induces a variational vector field X for Σ.

Assume that $v_t=\langle\nu_t,X\rangle$. Since this is a foliation, $v_t>0$ for small t. Set $\lambda(t)=H_{\Sigma_t}-h|_{\Sigma_t}$, by (5.1),

$$\lambda'(t)v_t^{-1}=-v_t^{-1}\Delta_{\Sigma_t}v_t-(\mathrm{Ric}(\nu_t)+|A_{\Sigma_t}|^2)-\nabla_{\nu_t}h.$$

Integrate, using Schoen-Yau's rewrite, integration by parts, we can find that

$$\lambda'(t)\leqslant 0, \text{ and hence } \lambda(t)\leqslant 0 \text{ for } t\geqslant 0.$$

It then follows from the first variation (3.2) that Ω_t is a minimiser for all small $t > 0$. Similar arguments applies to all small $t < 0$. Hence, we can carry all the rigidity analysis for Ω_0 to Ω_t.

5.2. **How to find a minimiser/issues.**

Recall that the existence theorem of a surface prescribed mean curvature requires that $H_{\partial_- M} \lneqq h$ on $\partial_- M$ and $H_{\partial_+ M} > h$ on $\partial_+ M$ to find a stable surface of prescribed mean curvature h.

Set $d(x) = \min\{\frac{\pi}{n}, -\frac{\pi}{n} + \operatorname{dist}(x, \partial_- M)\}$.

Since $\operatorname{width}(M, g) = \frac{2\pi}{n}$, $\operatorname{dist}(x, \partial_- M) \geqslant \frac{2\pi}{n}$ for any $x \in \partial_+ M$ (in some sense), so $d(\partial_+ M) = \frac{\pi}{n}$. But now there is no room for **tweak** for ϕ any more, because $|\nabla d| \leqslant 1$, $d(\partial_\pm M) = \pm\frac{\pi}{n}$, set $h = \eta(d(x))$ for $x \in M$ and $H_{\partial_\pm} M$ is in fact not even defined.

How do we find a surface of prescribed mean curvature h?

Still, we can take approximations h_ε of h such that $M_\varepsilon := h_\varepsilon^{-1}([-\frac{\pi}{n}, \frac{\pi}{n}])$ still lies a positive distance away from $\partial_\pm M$, this way we can obtain a surface Σ_ε of prescribed mean curvature h_ε.

There would create problems if we take a limit of Σ_ε! One is that Σ_ε disappears at infinity (to $\partial_\pm M$); the other is that although Σ_ε has nonempty intersection with some fixed compact set K, but the portion $\Sigma_\varepsilon \backslash K$ might also drift to $\partial_\pm M$ as well resulting a possible change in topology.

The second scene is better, and actually we can make good choices of the approximations h_ε such that $\Sigma_\varepsilon \cap K \neq \emptyset$ for all small ε.

Now we take a look at the stability of Σ_ε (dimension $n = 3$)

$$\int_{\Sigma_\varepsilon} (|\nabla^{\Sigma_\varepsilon} \phi|^2 + \tfrac{1}{2} R_{\Sigma_\varepsilon} \phi^2) \geqslant \frac{1}{2} \int_{\Sigma_\varepsilon} (R_M + \tfrac{3}{2} h_\varepsilon^2 + 2\nabla_{\nu_\varepsilon} h_\varepsilon) \phi^2$$

for all smooth ϕ, especially for $\phi = 1$.

In dimension 3, we know that the left hand side is less or equal to zero when $\phi = 1$ no matter where Σ_ε lies. If we can make

$$R_M + \tfrac{n}{n-1} h_\varepsilon^2 + 2\nabla_{\nu_\varepsilon} h_\varepsilon > 0, \text{ on } M_\varepsilon \backslash K,$$

then Σ_ε cannot lie entirely within $M_\varepsilon \backslash K$ according to (5.2), hence $\Sigma_\varepsilon \cap K$ is nonempty. As one may expect, $R_M + \frac{n}{n-1} h_\varepsilon^2 + 2\nabla_{\nu_\varepsilon} h_\varepsilon$ might not have a sign in K. You gain something, you lose something.

This strategy was first developped by G. Liu [Liu13], and since then, there were many versions of this strategy, especially in non-compact manifolds. We are using now a variant due to J. Zhu [Zhu21].

5.3. **Explicit construction of h_ε.**

Now let $a_0 = \frac{\pi}{n}$ (half of the band width), we can choose an odd smooth function $\beta(t) : [-a_0, a_0] \to \mathbb{R}$ such that

$$\begin{aligned} \beta(t) &> 0 \text{ on } (0, a_0], \\ \beta'(t) &> 0 \text{ on } [0, \tfrac{a_0}{2}), \\ \beta'(t) &< 0 \text{ on } (\tfrac{a_0}{2}, a_0]. \end{aligned}$$

See the shape of β as follows.

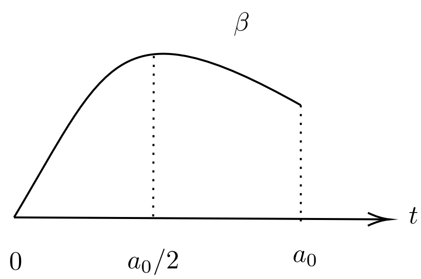

FIGURE 3. Shape of β on the positive t-axis.

Let η be the solution to the ODE

$$6 + \tfrac{3}{2}\eta^2 + 2\eta' = 0, \eta' < 0.$$

For every $\varepsilon > 0$, now we define a perturbation η_ε of η by setting $\eta_\varepsilon(t) = \eta(t+\varepsilon\beta(t))$ on a sub-interval $(-T_\varepsilon, T_\varepsilon)$ of $[-a_0, a_0]$ such that $\eta_\varepsilon(t) \to \pm\infty$ as $t = \mp T_\varepsilon$ (or $-a_0 \leqslant t + \varepsilon\beta(t) \leqslant a_0$), we can easily calculate that

$$\begin{aligned} 6 + \tfrac{3}{2}\eta_\varepsilon^2 + 2\eta_\varepsilon' =& 6 + \tfrac{3}{2}\eta_\varepsilon(t+\varepsilon\beta(t))^2 + 2\eta'(t+\varepsilon\beta(t))(1+\varepsilon\beta'(t)) \\ =& 2\varepsilon\beta'(t)\eta'(t+\varepsilon\beta(t)). \end{aligned}$$

Then we see

$$\begin{aligned} 6 + \tfrac{3}{2}\eta_\varepsilon^2 + 2\eta_\varepsilon' &> 0 \text{ if } \tfrac{a_0}{2} < |t| \leqslant T_\varepsilon, \\ 6 + \tfrac{3}{2}\eta_\varepsilon^2 + 2\eta_\varepsilon' &< 0 \text{ if } |t| < \tfrac{a_0}{2}. \end{aligned}$$

5.4. **Approximating surface Σ_ε.**

We set $h_\varepsilon(x) = \eta_\varepsilon(d(x))$, we set

$$M_\varepsilon = \{x : \ -\tfrac{\pi}{n} \leqslant d(x) + \varepsilon\beta(d(x)) \leqslant \tfrac{\pi}{n}\}.$$

And $\partial_\pm M_\varepsilon = \{d(x) + \varepsilon\beta(d(x)) = \pm\frac{\pi}{n}\}$. From the definition of T_ε, the boundries $\partial_\pm M_\varepsilon$ is prescisely where $d(x) = \pm T_\varepsilon$.

So $h(\partial_\pm M_\varepsilon) = \mp\infty$, hence, we can find a surface Σ_ε of prescribed mean curvature h_ε. From stability again,

$$\begin{aligned} \int_{\Sigma_\varepsilon} (|\nabla^{\Sigma_\varepsilon}\phi|^2 + \tfrac{1}{2}R_{\Sigma_\varepsilon}\phi^2) \geqslant& \tfrac{1}{2}\int_{\Sigma_\varepsilon} (R_M + \tfrac{3}{2}h_\varepsilon^2 + 2\nabla_{\nu_\varepsilon} h_\varepsilon)\phi^2 \\ \geqslant& \tfrac{1}{2}\int_{\Sigma_\varepsilon} (6 + \tfrac{3}{2}\eta_\varepsilon(d(x))^2 + 2\eta_\varepsilon'(d(x))). \end{aligned}$$

By construction, Σ_ε does not drift to $\partial_\pm M$, because it stays inside M_ε.

We set

$$K = \{x \in M : |d(x)| \leqslant \tfrac{a_0}{2}\}.$$

Also, Σ_ε can not lie entirely inside where $\frac{1}{2}a_0 \leqslant |d(x)| \leqslant T_\varepsilon$, that is, $\Sigma_\varepsilon \cap K \neq \emptyset$.

5.5. **What happens in the limit.**
Now we take limit of Σ_ε as $\varepsilon \to 0$. In the region $\{|d(x)| \leqslant a_0 - \delta\}$, $\delta > 0$, the limit of Σ_ε behaves well, since the prescribed mean curvature h_ε is uniformly bounded (also ∇h_ε is uniformly bounded as well). We can invoke some compactness theorem (up to a subsequence), the limit Σ is smooth in $\{-\delta < d(x) < \delta\}$ and hence in all $M \backslash \partial_\pm M$. Also, $\Sigma \cap K \neq \emptyset$.

But it still possible that some part of Σ drift to infinity. This is where the dimension come into play. We can take $\phi = 1$ and apply the Gauss-Bonnet theorem,

$$0 \geqslant \frac{1}{2} \int_{\Sigma_\varepsilon} (6 + \tfrac{3}{2}\eta_\varepsilon(d(x))^2 + 2\nabla_{\nu_\varepsilon} h_\varepsilon) \text{ on } \Sigma_\varepsilon.$$

Note that Σ_ε has genus higher than or equal to one. From simple re-arranging of the above, we have for Σ_ε that

$$\begin{aligned}
\int_{\Sigma_\varepsilon} (\langle \nu_\varepsilon, \nabla d\rangle - 1)\eta_\varepsilon' \circ d &\leqslant - \int_{\Sigma_\varepsilon} [6 + \tfrac{3}{2}(\eta_\varepsilon \circ d)^2 + \eta_\varepsilon' \circ d] \\
&= -\varepsilon \int_{\Sigma_\varepsilon} \beta'(d)\eta'(d + \varepsilon\beta(d)) \\
&= -\varepsilon \left(\int_{\Sigma_\varepsilon \cap K} + \int_{\Sigma_\varepsilon \backslash K} \right) \beta'(d)\eta'(d + \varepsilon\beta(d)) \\
&\leqslant -\varepsilon \int_{\Sigma_\varepsilon \cap K} \beta'(d)\eta'(d + \varepsilon\beta(d)) \\
&\leqslant C\varepsilon \operatorname{Area}(\Sigma_\varepsilon \cap K).
\end{aligned}$$

Note that $\eta_\varepsilon' \circ d < 0$ and $\langle \nu_\varepsilon, \nabla d\rangle - 1 \leqslant 0$, so the limit $\nu = \lim_\varepsilon \nu_\varepsilon$ satisfies $\langle \nu, \nabla d\rangle = 1$. Hence, Σ must be a level set of d.

Now we can carry the rigidity analysis as before.

5.6. **A remark on higher dimensions.**
In higher dimensions, you can still construct a Σ such that Σ always intersect a compact subset of M. However, in higher dimensions we do not have the extra control from the Gauss-Bonnet theorem.

6. Spectral scalar curvature bound

From now on, the notes would be a bit more focused on the research. Gromov asked the following question in [Gro19, Section 6.1.2]:

What are effects on the topology and/or metric geometry of a Riemannian manifold X played by the positivity of the

$$L_\gamma : f(x) \mapsto -\Delta f(x) + \gamma \cdot \mathrm{Sc}(X, x) f(x)$$

for a given constant $\gamma > 0$?

The spectral property of this operator is famously used in Chodosh-Li's work [CL23] in settling the stable Bernstein conjecture. Here we are concerned with a result which generalizes Gromov's band width estimate by replacing the scalar curvature with the condition that

$$\lambda_1(L_\gamma) > 0.$$

We also have some results if $\mathrm{Sc}(X, x)$ (in Gromov notation) is replaced by the Ricci curvature.

Theorem 6.1. *(joint with Yukai Sun 2025) For a Riemannian band $M^n = [-1, 1] \times \mathbb{T}^{n-1}$ $(3 \leqslant n \leqslant 7)$ with a smooth metric g, let u be a positive smooth function on $M \backslash \partial M$ with $u = 0$ on ∂M such that*

$$-\gamma \Delta_g u + \tfrac{1}{2} R_g u = \Lambda u,$$

where $\Lambda > 0$, $0 < \gamma < 2$. Then

$$\mathrm{width}(M, g) \leqslant \frac{\pi}{\sqrt{\Lambda}\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}}}.$$

Using similar techniques, we can prove a spectral version of Zhu's band width estimate with Ricci curvature lower bound (Theorem 4.4). The proof is basically identical to that of the scalar curvature bound except that we use

$$\mathrm{Ric}(\nu, \nu) + |A|^2 = \mathrm{Ric}(e_1, e_2) + \mathrm{Ric}(e_2, e_2) + H^2 - R_\Sigma,$$

instead of Schoen-Yau's rewrite.

7. Warped μ-bubble

For every $\gamma > 0$, we define for every open set Ω with smooth boundary

$$E(\Omega) = \int_{\partial^* \Omega} u^\gamma - \int (\chi_\Omega - \chi_{\Omega_0}) h u^\gamma$$

where $\Omega_- \subset \Omega_0 \subset \Omega_+$, where $u > 0$ and h is again a Lipschitz function.

We consider a smooth one-parameter family of deformations Ω_t of Ω, we can calculate the first variation of the functional E:

$$0 = \tfrac{\mathrm{d}}{\mathrm{d}t} E(\Omega_t)|_{t=0} = \int_{\partial \Omega} (H + \gamma u^{-1} u_\nu - h) u^\gamma \phi$$

where $\phi\nu$ is the variational vector field. Here,

$$u_\nu = \langle \nabla u, \nu \rangle.$$

We assume that $0 < \gamma < 2$.

Definition 7.1. *We say that Ω is a warped μ-bubble if $H + \gamma u^{-1} u_\nu - h = 0$ along $\partial\Omega$.*

Remark 7.2. More generally, one can also consider

$$E(\Omega)=\int_{\partial^*\Omega}u^\alpha-\int(\chi_\Omega-\chi_{\Omega_0})hu^\gamma,$$

with different exponents α and γ.

In calculating the second variation for E when Ω is a warped μ-bubble, we only have to calculate the variation $\nabla_{\phi\nu}(H+\gamma u^{-1}u_\nu-h)$. We have see $\nabla_{\phi\nu}H-h$. It remains to calculate $\nabla_{\phi\nu}(u^{-1}u_\nu)$. We see

$$\begin{aligned}&\nabla_{\phi\nu}(u^{-1}u_\nu)\\=&(-\phi u^{-2}u_\nu)u_\nu+u^{-1}\nabla_{\phi\nu}\langle\nabla u,\nu\rangle\\=&-\phi u^{-2}u_\nu^2+u^{-1}\langle\nabla_{\phi\nu}\nabla u,\nu\rangle+u^{-1}\langle\nabla u,\nabla_{\phi\nu}\nu\rangle\\=&-\phi u^{-2}u_\nu^2+u^{-1}\phi\operatorname{Hess}u(\nu,\nu)-u^{-1}\langle\nabla u,\nabla^\Sigma\phi\rangle.\end{aligned}$$

From the stability inequality,

$$\begin{aligned}0\leqslant&\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}E(\Omega_t)|_{t=0}\\=&\int_\Sigma[-\Delta_\Sigma\phi-|A|^2\phi-\operatorname{Ric}(\nu,\nu)\phi\\&\quad-\gamma u^{-2}u_\nu^2\phi+\gamma u^{-1}\phi\operatorname{Hess}u(\nu,\nu)-\gamma u^{-1}\langle\nabla_\Sigma u,\nabla_\Sigma\phi\rangle\\&\quad-h_\nu\phi]u^\gamma\phi\\=&\int_\Sigma[-\Delta_\Sigma\phi-|A|^2\phi-\operatorname{Ric}(\nu,\nu)\phi\\&\quad-\gamma u^{-2}u_\nu^2\phi+\gamma u^{-1}\phi(\Delta u-\Delta_\Sigma u-Hu_\nu)-\gamma u^{-1}\langle\nabla_\Sigma u,\nabla_\Sigma\phi\rangle\\&\quad-h_\nu\phi]u^\gamma\phi.\end{aligned}\tag{7.1}$$

We collect the following three terms

$$\int_\Sigma[\Delta_\Sigma\phi+\gamma u^{-1}\phi\Delta_\Sigma u+\gamma u^{-1}\langle\nabla_\Sigma u,\nabla_\Sigma\phi\rangle]u^\gamma\phi.$$

Observe that $u^\gamma\phi\Delta_\Sigma\phi$ comes from the variation of mean curvature, we want to make this term looks like $\psi\Delta_\Sigma\psi$, at least, in dimension three, we can then make $\psi=1$ and run the standard proofs.

This is done via setting $\phi = u^{-\gamma/2}\psi$. We have

$$\begin{aligned}
&\int_\Sigma [\Delta_\Sigma \phi + \gamma u^{-1}\phi\Delta_\Sigma u + \gamma u^{-1}\langle\nabla_\Sigma u, \nabla_\Sigma \phi\rangle] u^\gamma \phi \\
= &\int_{\partial\Omega} [\psi\Delta_\Sigma u^{-\gamma/2} + 2\langle\nabla_\Sigma u^{-\gamma/2}, \nabla_\Sigma\psi\rangle + u^{-\gamma/2}\Delta_\Sigma\psi + \gamma u^{-1-\gamma/2}\psi\Delta_\Sigma u \\
&\quad + \gamma u^{-1}\langle\nabla_\Sigma u, \nabla_\Sigma(u^{-\gamma/2}\psi)\rangle] u^{\gamma/2}\psi \\
= &\int_\Sigma \psi\Delta_\Sigma\psi \\
&\quad + \int_\Sigma [u^{\gamma/2}\psi^2\Delta_\Sigma u^{-\gamma/2} + 2u^{\gamma/2}\psi\langle\nabla_\Sigma u^{-\gamma/2}, \nabla_\Sigma\psi\rangle \\
&\qquad + \gamma u^{-1}\psi^2\Delta_\Sigma u + \gamma u^{\gamma/2-1}\psi\langle\nabla_\Sigma u, \nabla_\Sigma(u^{-\gamma/2}\psi)\rangle] \\
= &-\int_\Sigma |\nabla_\Sigma\psi|^2 \\
&\quad + \int_\Sigma [-\langle\nabla_\Sigma(u^{\gamma/2}\psi^2), \nabla_\Sigma u^{-\gamma/2}\rangle + 2u^{\gamma/2}\psi\langle\nabla_\Sigma u^{-\gamma/2}, \nabla_\Sigma\psi\rangle \\
&\qquad - \gamma\langle\nabla_\Sigma(u^{-1}\psi^2), \nabla_\Sigma u\rangle + \gamma u^{\gamma/2-1}\psi\langle\nabla_\Sigma u, \nabla_\Sigma(u^{-\gamma/2}\psi)\rangle].
\end{aligned}$$

In the last line we have used integration by parts on the two terms containing $\Delta_\Sigma u$ and the term containing $\Delta_\Sigma\psi$. By a direct calculation, we conclude that

$$\begin{aligned}
&\int_\Sigma [\Delta_\Sigma \phi + \gamma u^{-1}\phi\Delta_\Sigma u + \gamma u^{-1}\langle\nabla_\Sigma u, \nabla_\Sigma\phi\rangle] u^\gamma\phi \\
= &-\int_\Sigma |\nabla_\Sigma\psi|^2 + \int_\Sigma [-\gamma\psi\langle\nabla_\Sigma w, \nabla_\Sigma\psi\rangle + (\gamma - \tfrac{\gamma^2}{4})\psi^2|\nabla_\Sigma w|^2]
\end{aligned}$$

where we have set $w = \log u$. Using the above, $\phi = u^{-\gamma/2}\psi$ and $w = \log u$ in (7.1), we see

$$\begin{aligned}
0 \leqslant &\int_\Sigma |\nabla_\Sigma\psi|^2 + \int_\Sigma [\gamma\psi\langle\nabla_\Sigma w, \nabla_\Sigma\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_\Sigma w|^2] \\
&+ \int_\Sigma [\gamma u^{-1}\Delta u - (|A|^2 + \mathrm{Ric}(\nu))]\psi^2 \\
&- \int_\Sigma [\gamma H w_\nu + h_\nu + \gamma w_\nu^2]\psi^2. \qquad (7.2)
\end{aligned}$$

7.1. **Considering the spectral condition.** We assume that

$$-\gamma\Delta u + \tfrac{1}{2}R_g u = \Lambda u$$

where $\Lambda > 0$, $0 < \gamma < 2$, $u > 0$.

Using Schoen-Yau rewrite,

$$|A|^2 + \mathrm{Ric}(\nu) = \tfrac{1}{2}(R_g - R_\Sigma + |A|^2 + H^2)$$

and $|A|^2 \geqslant H^2/(n-1)$, we have

$$\begin{aligned}&\int_\Sigma [\gamma u^{-1}\Delta u - (|A|^2 + \mathrm{Ric}(\nu))]\psi^2\\ \leqslant &\int_\Sigma \left[\gamma u^{-1}\Delta u - \tfrac{1}{2}(R_g - R_\Sigma + \tfrac{n}{n-1}H^2)\right]\psi^2\\ = &\int_\Sigma \left(\tfrac{1}{2}R_\Sigma - \tfrac{n}{2(n-1)}H^2 - \Lambda\right)\psi^2.\end{aligned}$$

Using the above and that $H = -\gamma w_\nu + h$ in (7.2), we arrive

$$\begin{aligned}0 \leqslant &\int_\Sigma |\nabla_\Sigma\psi|^2 + \int_\Sigma [\gamma\psi\langle\nabla_{\partial\Omega}w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_\Sigma w|^2]\\ &+ \int_\Sigma [\tfrac{1}{2}R_\Sigma - \tfrac{n}{2(n-1)}(-\gamma w_\nu + h)^2 - \Lambda]\psi^2\\ &- \int_\Sigma [\gamma(-\gamma w_\nu + h)w_\nu + h_\nu + \gamma w_\nu^2]\psi^2\\ = &\int_\Sigma |\nabla_\Sigma\psi|^2 + \tfrac{1}{2}R_\Sigma\psi^2 + \int_\Sigma [\gamma\psi\langle\nabla_\Sigma w, \nabla_\Sigma\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_\Sigma w|^2]\\ &- \int_\Sigma \left[(\tfrac{n}{2(n-1)}\gamma^2 - \gamma^2 + \gamma)w_\nu^2 - \tfrac{1}{n-1}\gamma h w_\nu + \tfrac{n}{2(n-1)}h^2 - |\nabla h| + \Lambda\right]\psi^2.\end{aligned}$$

Since $\gamma^2/4 - \gamma < 0$, so by Cauchy-Schwarz inequality,

$$\int_\Sigma [\gamma\psi\langle\nabla_\Sigma w, \nabla_\Sigma\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_\Sigma w|^2] \leqslant \tfrac{1}{4}\gamma(1 - \tfrac{\gamma}{4})^{-1}\int_\Sigma |\nabla_\Sigma\psi|^2.$$

and by Cauchy-Schwarz inequality,

$$\begin{aligned}&(\tfrac{n}{2(n-1)}\gamma^2 - \gamma^2 + \gamma)w_\nu^2 - \tfrac{1}{n-1}\gamma h w_\nu + \tfrac{n}{2(n-1)}h^2\\ \geqslant &\left[-\frac{\gamma^2}{4(\frac{n}{2(n-1)}\gamma^2 - \gamma^2 + \gamma)(n-1)^2} + \tfrac{n}{2(n-1)}\right]h^2\\ = &\frac{-n\gamma + \gamma + 2n}{4(\frac{n}{2(n-1)}\gamma - \gamma + 1)(n-1)}h^2,\end{aligned}$$

which is positive by the assumption $0 < \gamma < 2$. Therefore,

$$\begin{aligned}0 \leqslant &(1 + \tfrac{1}{4}\gamma(1 - \gamma/4)^{-1})\int_\Sigma |\nabla_\Sigma\psi|^2 + \tfrac{1}{2}\int_\Sigma R_\Sigma\psi^2\\ &- \int_\Sigma \left[\frac{-n\gamma + \gamma + 2n}{4(\frac{n}{2(n-1)}\gamma - \gamma + 1)(n-1)}h^2 - |\nabla h| + \Lambda\right]h^2.\end{aligned}$$

7.2. **A remark on the rigidity analysis.** Our final form (or consequence) of the stability of the weighted μ-bubble is similar to the un-weighted version of the μ-bubble. In dimension three, the rigidity is based on the study of

$$H + \gamma u^{-1}u_\nu - h.$$

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