RIGIDITY AND WEAK NOTIONS OF SPECTRAL SCALAR CURVATURE

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Abstract. We give a weak formulation of spectral scalar curvature bounded from below via establishing a dihedral rigidity result. Given some special convex polyhedron, if another metric are of non-negative spectral scalar curvature in the interior, with weighted mean-convex faces, and with its dihedral angles less than or equal to their flat polyhedral model everywhere along the edges, then the metric must be flat. This is motivated by Gromov's definition of a weak notion of non-negative scalar curvature.

1. Introduction

Gromov [Gro14] initiated the study of scalar curvature for C^0 metrics and he gave the following definition.

Definition 1.1. *Given a continuous metric g on M, we say that $R_g \geqslant 0$ in the C^0 sense at a point $p \in M$ if around p there does not exist a cube such that its face is strictly mean-convex and the dihedral angles are acute.*

It can be equivalently formulated in the dihedral rigidity conjecture for cubes [Gro14] which states that a Riemannian metric on a Euclidean convex polyhedron of non-negative scalar curvature, with weakly mean-convex faces and non-obtuse angles must be flat. There is an alternative definition using the Ricci flow [Bur19].

The dihedral rigidity conjecture has been proved in several cases: Li confirmed the conjecture for Euclidean frusta and pyramids [Li20a] in dimension 3 with some additional assumptions as well as for n-prisms [Li24], up to dimension seven. Brendle [Bre24] and Brendle-Wang [BW23] confirmed the Euclidean conjecture with additional hypothesis on angles. The most general case was claimed by Wang-Xie-Yu [WXY22]. There is also a hyperbolic version of the conjecture for parabolic cubes, see Gromov [Gro14], Li [Li20b], Wang-Xie [WX23]; for the hyperbolic version modeled on polyhedra in the upper half-space model of the hyperbolic space, see Chai-Wang [CW24b] and Chai-Wan [CW24a]. The dihedral rigidity can be put in a broader context of scalar curvature rigidity, among which the earliest of such results are the Geroch conjecture [SY79a], [GL83] and positive mass theorems

[SY79b], [Wit81]. In these early works, two major techniques of scalar curvature geometry, minimal surface and spinors were developed.

The spectral scalar curvature is defined as the first eigenvalue of an elliptic operator which is the sum of the Laplacian and the scalar curvature. We denote the first eigenvalue by $\lambda_1(-\gamma\Delta_g+\frac{1}{2}R_g)$. Here, R_g is the scalar curvature and Δ_g is the Laplacian-Beltrami operator. We only consider the case $\gamma>0$ and the coefficient $\frac{1}{2}$ on R_g is for convenience. For a closed manifold (M,g), $R_g\geqslant 0$ obviously implies that $\lambda_1(-\gamma\Delta_g+\frac{1}{2}R_g)\geqslant 0$, hence, $\lambda_1(-\gamma\Delta_g+\frac{1}{2}R_g)\geqslant 0$ is a weaker condition. Here, we give a slight different version more suitable for manifolds with boundary.

Definition 1.2. *Let (M,g) be a Riemannian manifold and u be a positive function, we call*

$$-\gamma u^{-1}\Delta_g u+\tfrac{1}{2}R_g \tag{1.1}$$

the spectral scalar curvature. Given an oriented hypersurface Σ with a chosen unit normal N, we call $H+\gamma u^{-1}\partial_N u$ the weighted mean curvature. We will explicitly indicate the dependence on γ and u if needed. Here, $H=\operatorname{div}_\Sigma N$ is the mean curvature of Σ in (M,g).

In a closed manifold, $-\gamma u^{-1}\Delta_g u+\frac{1}{2}R_g\geqslant 0$, $u>0$ is easily seen to be equivalent to that the first eigenvalue of the operator $-\gamma\Delta_g+\frac{1}{2}R_g$ is non-negative.

Analogous to Gromov's definition [Gro14] (i.e., Definition 1.1) of non-negative scalar curvature for C^0 metrics, we have the definition: given a continuous positive function u and a continuous metric g on M, we say that $-\gamma u^{-1}\Delta_g u+\frac{1}{2}R_g\geqslant 0$ in the C^0 sense at a point $p\in M$ if around p there does <u>not</u> exist a cube such that its face is strictly weighted mean-convex and the dihedral angles are acute.

This is just a definition by simply replacing the scalar curvature and the mean curvature in by their spectral, or weighted counterparts. Similar statements can be made for arbitrary convex polyhedra among which we find it more convenient to state for the cubes. We would like to formulate $-\gamma u^{-1}\Delta_g u+\frac{1}{2}R_g\geqslant\Lambda$ in the C^0 sense as well, where Λ is a negative constant and the case $\gamma=0$ was conjectured by Gromov [Gro14] (cf. [CW24b, Conjecture 1.1]).

Most generally, we have the following Gromov dihedral rigidity conjecture for the spectral scalar curvature.

Conjecture 1.3. *Let $0\leqslant\gamma<\frac{2n}{n-1}$, $\Lambda\leqslant 0$ and*

$$\beta=\frac{\sqrt{-2\Lambda}}{\sqrt{2(n-1)-(n-2)\gamma}\sqrt{2n-(n-1)\gamma}},\ \alpha=(2-\gamma)\beta,\ h=(n-1)\alpha+\beta\gamma.$$

Let Ω be a convex polyhedron in the Euclidean space with a distinguished unit vector N_0, N_i be the Euclidean unit normal vector of the face F_i of Ω pointing *outward of* Ω. *Let* g *be a metric and* u *be a function defined on a neighborhood of* $\bar{\Omega}$. *If* $u > 0$ *on* $\bar{\Omega}$ *and* g *satisfies*

$$-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g \geqslant \Lambda \text{ in } \Omega,$$

and

$$H_{F_i} + \gamma u^{-1}\tfrac{\partial u}{\partial \nu_i} \geqslant -h\langle N_0, N_i\rangle =: -h\cos\bar{\theta}_i \text{ along every } F_i,$$

and the dihedral angles $\alpha_{i,j} \leqslant \bar{\alpha}_{i,j}$ *along the edge* $F_i \cap F_j$, *then* (Ω, g) *is isometric to some polyhedron in the upper half-space model* $\bar{g} = \frac{1}{\alpha^2 t^2}(\mathrm{d}t^2 + g_{\mathbb{R}^{n-1}})$ *and* u *is a constant multiple of* $\bar{u} = t^{-\beta/\alpha}$.

Remark 1.4. *We call* Ω *with the Riemannian metric* g *a Riemannian polyhedron, and* Ω *with the flat metric a reference polyhedron or just a reference which we denote by* P.

Remark 1.5. *In horocyclic coordinates,* $\bar{g} = \mathrm{d}s^2 + e^{2\alpha s}g_{\mathbb{R}^{n-1}}$ *and* $u = e^{\beta s}$ $(t = \alpha^{-1}e^{-\alpha s}$, $s = \frac{\log(\alpha t)}{-\alpha})$. *The metric* $\bar{g}$ *and the function* $\bar{u}$ *were found by the author and Yukai Sun (Henan University, China; in preparation) where* $g_{\mathbb{R}^{n-1}}$ *is replaced by the flat metric on the torus.*

Remark 1.6. *There is some freedom to consider the condition*

$$-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g + cu^{-2}|\nabla_g u|^2 \geqslant \Lambda \tag{1.2}$$

with suitable range of c *and* γ, *we leave the reader to deduce Proposition 2.4 for* (1.2) *and formulate corresponding Conjecture 1.3.*

1.1. **Frustum and pyramid.**

With some additional conditions, we confirm Conjecture 1.3 for two types of polyhedra which we now describe.

Definition 1.7. *Let* $k \geqslant 3$ *be an integer,* $B_1 \subset \{x^3 = 0\}$ *and* $B_2 \subset \{x^3 = 1\}$ *be two similar* k*-polygons whose corresponding edges are parallel. We call the set* $\{tp + (1-t)q : \ p \in B_1, q \in B_2\}$ *a* (B_1, B_2)*-frustum,* B_1 *its base face and* B_2 *its top face. (Most of the time, we just call the* (B_1, B_2)*-frustum a frustum.)*

A frustum (*plural*, *frusta*) is a portion of a solid that lies between two parallel planes cutting the solid. It is a solid itself.

Definition 1.8. *Let* $k \geqslant 3$ *be an integer,* $B \subset \{x^3 = 0\}$ *be a* k*-polygon and* $p \in \{x^3 = 1\}$. *We call the set* $\{tp + (1-t)q : \ q \in B\}$ *a* (B, p)*-pyramid (or just pyramid if the references to* B *and* p *are clear),* p *the apex of the pyramid and* B *the base.*

Remark 1.9. *The frustum defined here is what Li* [Li20a, Definition 1.3] *called a* (B_1, B_2)*-prism and the pyramid is what he called a* (B, p)*-cone, see also* [CW24b, Definition 1.3].

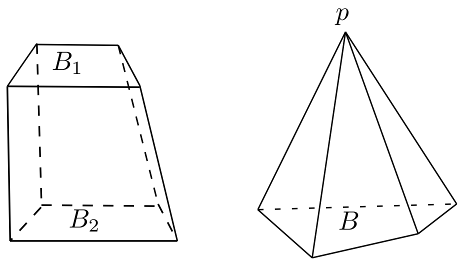

FIGURE 1.1. A frustum and a pyramid.

Theorem 1.10. *Conjecture 1.3 holds in dimension 3 (in which case $0 \leqslant \gamma < 3$) for frustum with the additional assumption that the bottom face of the reference is normal to N_0, and for any neighboring faces F_j and F_{j+1}, the angle condition*

$$|\pi - (\alpha_j + \alpha_{j+1})| < \alpha_{j,j+1}$$

holds. Here, α_j is the dihedral angle of F_j with the base face and $\alpha_{j,j+1}$ is the dihedral angle formed by F_j and F_{j+1}. The same conclusion holds for the pyramid if the pyramid has an isometric tangent cone at the apex to the tangent cone of its Euclidean model or it is a tetrahedron.

Remark 1.11. *In dimension 3, $\beta = \frac{\sqrt{-\Lambda}}{\sqrt{4-\gamma}\sqrt{3-\gamma}}$, $\alpha = \frac{(2-\gamma)\sqrt{-\Lambda}}{\sqrt{4-\gamma}\sqrt{3-\gamma}}$ and $h = \sqrt{\frac{4-\gamma}{3-\gamma}}$. When $\gamma = 2$, $\alpha = 0$ and it will cause a minor issue which can be resolved by changing coordinates, see Remark 1.5.*

Theorem 1.10 is the natural spectral analog of [Li20a] and [CW24b], in fact, the polyhedra considered here are precisely those already considered in [Li20a] and [CW24b], also the techniques are quite similar. Theorem 1.10 serves as a starting point for future work on generalizations in higher dimensions and to more types of polyhedra.

A simple consequence of Theorem 1.10 is that the weak notions of spectral scalar curvature $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g \geqslant \Lambda$ for some constant $\Lambda \leqslant 0$ makes sense. Hence, we answered the question of formulation of $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g$ bounded from below in the C^0 sense. It might be interesting to explore this definition using the Ricci flow, cf. [Bur19], also, it is an interesting question to explore the preservation of the lower bound $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g \geqslant \Lambda$ with respect to the connvergence of C^0 metrics, see [Gro14]. It is also an interesting question to look for analog of [CW24a], [CW23] and [KY24], which will be addressed in a future work.

The article is organized as follows:

In Section 2, we introduce the capillary warped μ-functional, calculate its first and second variation, in particular, we relate the spectral curvature condition (1.1) to the second variation. In Sections 3 and 4, we prove the frustum and pyramid case of Theorem 1.10.

2. Capillary warped μ-bubble

In this section, we study a capillary version of the warped μ-bubble, in particular, we give the related geometric functional, calculate its first and second variations. Most importantly, we relate the second variation with the spectral scalar curvature.

We setup some notations: Let Σ be a surface which meet Ω transversely, and E be a connected component of $\Omega\backslash\Sigma$. Let

- N be the unit normal of Σ in Ω,
- X be the unit outward normal of $\partial\Omega$ in Ω,
- ν the unit outward normal of $\partial\Sigma$ in Σ,
- η be the unit normal of $\partial\Sigma$ in Ω which points outward of $\partial E\cap\partial\Omega$,
- and $\theta\in(0,\pi)$ be the contact angle between Σ and $\partial\Omega$ defined by $\cos\theta = \langle X, N\rangle$.

See Figure 2.1.

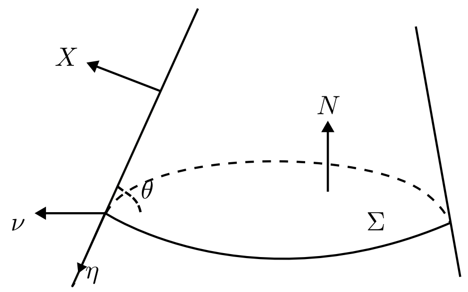

Figure 2.1. Labelling of various normal vectors.

2.1. **Capillary warped μ-bubble.** We define the warped μ-bubble functional

$$\mathcal{F}(E) = \int_{\partial E\cap \operatorname{int} M} u^\gamma \mathrm{d}\mathcal{H}^2 - \int_E u^\gamma \mu \mathrm{d}\mathcal{H}^3 - \sum_j \int_{\partial E\cap F_j} u^\gamma \cos\bar{\theta}_j \mathrm{d}\mathcal{H}^2,\ E\in\mathcal{E},$$

where $\mathcal{E}$ is defined as the set of the contractible open sets $E'\subset\mathcal{E}'$ and $\mathcal{E}'$ is given by

$$\mathcal{E}' := \begin{cases} \{E\subset M : p\in E, E\cap B=\emptyset\},\ P \text{ is a pyramid,} \\ \{E\in M :\ B_1\subset E,\ E\cap B_2=\emptyset\},\ P \text{ is a frustum.}\end{cases}$$

And j run through all indices such that $\{F_j\}$ run through all side faces. We consider the variational problem

$$I = \inf\{\mathcal{F}(E) : E \in \mathcal{E}\}. \tag{2.1}$$

For every $C^{1,\alpha}$ surface Σ and a smooth family of diffeomorphisms $\phi_t : \Sigma \to \Omega$ such that $\phi_t(\partial\Sigma) \subset \partial\Omega$, $\Sigma_t = \phi_t(\Sigma)$ such that $\Sigma_0 = \Sigma$, we define

$$\mathcal{A}(t) = \int_{\Sigma_t} u^\gamma \mathrm{d}\mathcal{H}^2 + \int_{E_t} \mu u^\gamma \mathrm{d}\mathcal{H}^3 - \sum_j \int_{\partial E_t \cap F_j} \cos\bar{\theta}_j u^\gamma \mathrm{d}\mathcal{H}^2, \tag{2.2}$$

where $E_t \in \mathcal{E}$ is the connected component of $\Omega\backslash\Sigma$ closer to the vertex or the top face. The first variation is given by

$$\mathcal{A}'(0) = \int_\Sigma (\gamma\langle\nabla u, Y\rangle u^{\gamma-1} + u^\gamma \operatorname{div}_\Sigma Y - \mu u^\gamma \langle Y, N\rangle) - \sum_j \int_{\partial\Sigma\cap F_j} \cos\bar{\theta}_j u^\gamma \langle Y, \eta\rangle,$$

here $Y = \frac{\partial}{\partial t}\phi_t$ is the vector field associated with ϕ_t. We decompose $Y = Y^\top + Y^\perp$ where $Y^\perp$ is the component normal to Σ. Let $\phi = \langle Y, N\rangle$, then $Y^\perp = \phi N$, $\langle\nabla u, Y\rangle = \langle\nabla u, Y^\top\rangle + \phi\langle\nabla u, N\rangle$, $\operatorname{div}_\Sigma Y = \phi H + \operatorname{div}_\Sigma Y^\top$ and

$$\mathcal{A}'(0) = \int_\Sigma \operatorname{div}_\Sigma(u^\gamma Y^\top) + \int_\Sigma (H + \gamma u^{-1}u_N - \mu)\phi - \sum_j \int_{\partial\Sigma\cap F_j} \cos\bar{\theta}_j u^\gamma \langle Y, \eta\rangle.$$

We set $\cos\theta = \langle X, N\rangle$, then $\langle Y, \eta\rangle = -f/\sin\theta$ and $\langle Y, \nu\rangle$. Integration by parts for the first term yields

$$\begin{aligned} \mathcal{A}'(0) &= \int_\Sigma (H + \gamma u^{-1}u_N - \mu)\phi + \sum_j \int_{\partial\Sigma\cap F_j} u^\gamma(\langle Y, \nu\rangle - \cos\bar{\theta}_j\langle Y, \eta\rangle) \\ &= \int_\Sigma (H + \gamma u^{-1}u_N - \mu)\phi - \sum_j \int_{\partial\Sigma\cap F_j} \tfrac{u^\gamma\phi}{\sin\theta}(\cos\theta_j - \cos\bar{\theta}_j). \end{aligned} \tag{2.3}$$

Definition 2.1. *We say that Σ is a capillary warped μ-bubble if $\mathcal{A}'(0) = 0$ for all E_t. Equivalently,*

$$\begin{aligned} H + \gamma u^{-1}u_N - \mu = 0 &\text{ in } \Sigma, \\ \langle X, N\rangle = \cos\bar{\theta}_j &\text{ along } \partial\Sigma\cap F_j. \end{aligned}$$

Definition 2.2. *We say that Σ is a stable capillary warped μ-bubble if $\mathcal{A}''(0) \geqslant 0$ for all E_t. The inequality $\mathcal{A}''(0) \geqslant 0$ is called the stability inequality.*

Now we calculate the second derivatives of $\mathcal{A}$ if Σ is a capillary warped μ-bubble.

Lemma 2.3. *If Σ is a capillary warped μ-bubble, then*

$$\mathcal{A}''(0) = \int_\Sigma u^\gamma\phi(-\Delta_\Sigma\phi - \gamma u^{-1}\phi\Delta_\Sigma u - \gamma u^{-1}\langle\nabla_\Sigma u, \nabla_\Sigma\phi\rangle + Z\phi) + \sum_j \int_{\partial\Sigma\cap F_j} u^\gamma\phi(\tfrac{\partial\phi}{\partial\nu} - q\phi),$$

where $w = \log u$,

$$Z := -|A|^2 - \mathrm{Ric}(N) - \gamma w_N^2 \phi + \gamma u^{-1} \Delta_g u - \gamma H w_N,$$

and

$$q = \tfrac{1}{\sin \bar{\theta}_j} A_{\partial M}(\eta, \eta) - \cot \bar{\gamma} A(\nu, \nu). \tag{2.4}$$

Proof. Since Σ is a capillary warped μ-bubble, we have

$$\mathcal{A}''(0) = \int_\Sigma \delta_Y (H + \gamma u^{-1} u_N - \mu)\phi - \sum_j \int_{\partial\Sigma \cap F_j} \tfrac{u^\gamma \phi}{\sin \theta_j} \delta_Y (\cos \theta_j - \cos \bar{\theta}_j). \tag{2.5}$$

It suffices to only compute the first variation of $H + \gamma w_N - \mu$ and $\cos\theta - \cos\bar{\theta}$. First,

$$\begin{aligned} &\delta_Y (H + \gamma w_N - \mu) = \delta_Y (H + \gamma w_N) \\ =& - \Delta_\Sigma \phi - (|A|^2 + \mathrm{Ric}(N))\phi - \gamma u^{-2} u_N^2 \phi + \gamma u^{-1} \phi \nabla^2_{NN} u - \gamma u^{-1} \langle \nabla_\Sigma u, \nabla_\Sigma \phi \rangle + \nabla_{Y^\top} (H + \gamma w_N) \\ =& - \Delta_\Sigma \phi - \gamma u^{-1} \phi \Delta_\Sigma u - \gamma u^{-1} \langle \nabla_\Sigma u, \nabla_\Sigma \phi \rangle + Z\phi \end{aligned} \tag{2.6}$$

where we have used $\nabla^2_{NN} u = \Delta_g u - \Delta_\Sigma u - H u_N$ and $Y^\top$ is the component of Y tangential to Σ. Note that $\nabla_{Y^\top}(H + \gamma w_N) = 0$ since $H + \gamma w_N$ is constant along Σ. Using [RS97, Appendix], we find that

$$\delta_Y \cos \theta_j = - \sin \theta_j \tfrac{\partial \phi}{\partial \nu} + q\phi + \nabla_{Y^{\partial\Sigma}} \cos \theta_j, \tag{2.7}$$

where $Y^{\partial\Sigma}$ is the component of Y tangential to $\partial\Sigma$. Inserting (2.6) and (2.7) into (2.5) finishes the proof. $\square$

2.2. **Rewrite.** Now we give the crucial rewrite which is vital to our proof.

Proposition 2.4. *Let* $\psi = u^{\gamma/2}\phi$, *then*

$$\begin{aligned} &\mathcal{A}''(0) \\ =& \tfrac{4}{4-\gamma} \int_\Sigma |\nabla_\Sigma \psi|^2 - (1 - \tfrac{\gamma}{4})\gamma \int_\Sigma \left| \psi \nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)} \nabla_\Sigma \psi \right|^2 \\ &+ \left(\tfrac{1}{2} \int_\Sigma R_\Sigma \psi^2 + \sum_j \int_{\partial\Sigma \cap F_j} \kappa_{\partial\Sigma} \psi^2 \right) - \left(\int_\Sigma W \psi^2 + \sum_j \int_{\partial\Sigma} \tfrac{1}{\sin \bar{\theta}_j} (H_{\partial M} + \gamma \partial_X w - \mu \cos \bar{\theta}_j) \psi^2 \right), \end{aligned} \tag{2.8}$$

where

$$W := \Big(\tfrac{3-\gamma}{4-\gamma}\mu^2 + (-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g)\Big) + \tfrac{1}{2}(|A|^2 - \tfrac{1}{2}H^2) + (1-\tfrac{1}{4}\gamma)(w_N - \tfrac{1}{4-\gamma}\mu)^2.$$

Proof. It follows from Schoen-Yau's rewrite

$$|A|^2 + \mathrm{Ric}(N) = \tfrac{1}{2}R_g - \tfrac{1}{2}R_\Sigma + \tfrac{1}{2}|A|^2 + \tfrac{1}{2}H^2,$$

of the twicely constracted Gauss equation, $H = -\gamma w_N + \mu$ and suitable regrouping that

$$Z = -W + \tfrac{1}{2}R_\Sigma. \tag{2.9}$$

By a direct calculation using $\psi = u^{\gamma/2}\phi$,

$$\begin{aligned}
& u^\gamma\phi(-\Delta_\Sigma\phi - \gamma u^{-1}\phi\Delta_\Sigma u - \gamma u^{-1}\langle\nabla_\Sigma u, \nabla_\Sigma\phi\rangle) \\
= & -\psi\Delta_\Sigma\psi + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_\Sigma w|^2 - \tfrac{\gamma}{2}\psi^2\Delta_\Sigma w \\
= & |\nabla_\Sigma\psi|^2 + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_\Sigma w|^2 + \gamma\langle\nabla_\Sigma w, \nabla_\Sigma\psi\rangle \\
& \quad - \mathrm{div}_\Sigma(\psi\nabla_\Sigma\psi) - \mathrm{div}(\tfrac{\gamma}{2}\psi^2\nabla_\Sigma w) \\
= & \tfrac{4}{4-\gamma}|\nabla_\Sigma\psi|^2 - (1-\tfrac{\gamma}{4})\gamma\left|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\nabla_\Sigma\psi\right|^2 \\
& \quad - \mathrm{div}_\Sigma(\psi\nabla_\Sigma\psi) - \mathrm{div}_\Sigma(\tfrac{\gamma}{2}\psi^2\nabla_\Sigma w).
\end{aligned} \tag{2.10}$$

Integration of the above omitting the divergence term, and (2.9) show the terms of the integration over Σ in (2.8).

It is left to deal with the boundary integration. We have collecting the divergence term in (2.10) and using the divergence theorem that

$$\begin{aligned}
& \sum_j \int_{\partial\Sigma\cap F_j} u^\gamma\phi(\tfrac{\partial\phi}{\partial\nu} - q\phi) - \int_\Sigma (\mathrm{div}_\Sigma(\psi\nabla_\Sigma\psi) + \mathrm{div}_\Sigma(\tfrac{\gamma}{2}\psi^2\nabla_\Sigma w)) \\
= & \sum_j \int_{\partial\Sigma\cap F_j} (e^{\gamma w/2}\psi\tfrac{\partial(e^{-\gamma w/2}\psi)}{\partial\nu} - \psi\partial_\nu\psi - \tfrac{\gamma}{2}\psi^2\partial_\nu w - q\psi^2) \\
= & -\sum_j \int_{\partial\Sigma\cap F_j} (\gamma\partial_\nu w + q)\psi^2.
\end{aligned}$$

Recall that the rewrite (see [RS97, Lemma 3.1] or [Li20a, (4.13)])

$$q = \tfrac{1}{\sin\bar\theta_j}A_{\partial M}(\eta,\eta) - \cot\bar\theta_j A(\nu,\nu) = -H\cot\bar\theta_j + \tfrac{H_{\partial M}}{\sin\bar\theta_j} - \kappa,$$

and using $H = -\gamma w_N + \mu$, we have

$$\begin{aligned} &\gamma \partial_\nu w + q \\ =&\gamma \partial_\nu w + (-(\mu - \gamma \partial_N w) \cot \bar{\theta}_j + \tfrac{H_{\partial M}}{\sin \bar{\theta}_j} - \kappa) \\ =& - \kappa + \tfrac{1}{\sin \bar{\theta}_j} (H_{\partial M} + \gamma (\cos \bar{\theta} \partial_N w + \sin \bar{\theta} \partial_\nu w) - \mu \cos \bar{\theta}_j) \\ =& - \kappa + \tfrac{1}{\sin \bar{\theta}_j} (H_{\partial M} + \gamma \partial_X w - \mu \cos \theta_j). \end{aligned} \tag{2.11}$$

This finishes the proof. □

Remark 2.5. *When Σ only satisfies the contact angle condition, but $H + \gamma u^{-1} u_N - \mu =: \tilde{H}$ might not vanish along Σ, then Z and W satisfy*

$$Z = -W + \tfrac{1}{2} R_\Sigma - \tfrac{3}{4} \tilde{H}^2 - \tfrac{1}{2} \tilde{H} (3\mu - \gamma w_N) \tag{2.12}$$

instead of (2.9). *And instead of* (2.11),

$$\gamma \partial_\nu w + q = -\kappa + \tfrac{1}{\sin \theta_j} (H_{\partial M} + \gamma \partial_X w - \mu \cos \theta_j) - \tilde{H} \cot \theta_j. \tag{2.13}$$

3. Rigidity of frustums

In this section, we prove the rigidity of frusta.

3.1. **Infinitesimal rigidity.**

Lemma 3.1. *Let E be a stable capillary μ-bubble in (Ω, g) specified in Theorem 1.10, then $\Sigma := \partial E \cap \operatorname{int} \Omega$ is infinitesimally rigid, that is,*

$$\begin{gathered} \nabla_\Sigma w = 0, \ |A|^2 - \tfrac{1}{2} H^2 = 0, \ - \gamma u^{-1} \Delta_g u + \tfrac{1}{2} R_g = \Lambda, \\ w_N = - \tfrac{1}{4 - \gamma} h, \ R_\Sigma = 0 \ \text{in } \Sigma, \\ H_{\partial M} + \gamma \partial_X w = -h \cos \bar{\theta}_j, \ \kappa_{\partial \Sigma} = 0 \ \text{along } \partial \Sigma \cap F_j, \\ \alpha_j = \bar{\alpha}_j \ \text{at } x \in \partial \Sigma \cap E_j. \end{gathered}$$

Proof. Putting $\psi = 1$ in (2.8) yields

$$\begin{aligned} 0 \leqslant & - (1 - \tfrac{\gamma}{4}) \gamma \int_\Sigma |\nabla_\Sigma w|^2 \\ & + \left(\tfrac{1}{2} \int_\Sigma R_\Sigma + \sum_j \int_{\partial \Sigma \cap F_j} \kappa_{\partial \Sigma} \right) - \left(\int_\Sigma W + \sum_j \int_{\partial \Sigma} \tfrac{1}{\sin \theta_j} (H_{\partial M} + \gamma \partial_X w + h \cos \bar{\theta}_j) \right). \end{aligned} \tag{3.1}$$

First, by the Gauss-Bonnet theorem,

$$\tfrac{1}{2}\int_\Sigma R_\Sigma + \sum_j \int_{\partial\Sigma\cap F_j} \kappa_{\partial\Sigma} + \sum_j (\pi - \alpha_j) = 2\pi\chi(\Sigma),$$

where α_j are the interior turning angles of $\partial\Sigma$ at a non-smooth point of $\partial\Sigma$. Since $\alpha_j \leqslant \bar{\alpha}_j$, and $\sum_j(\pi - \bar{\alpha}_j) = 2\pi$, so

$$\tfrac{1}{2}\int_\Sigma R_\Sigma + \sum_j \int_{\partial\Sigma\cap F_j} \kappa_{\partial\Sigma} \leqslant 0. \tag{3.2}$$

Then we check that $W \geqslant 0$ due to $|A|^2 - H^2/2 \geqslant 0$, $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g \geqslant \Lambda$ and $\frac{3-\gamma}{4-\gamma}h^2 + \Lambda = 0$. Also, $H + \gamma\partial_X w \geqslant -h\cos\bar{\theta}_j$ by the assumptions. So the inequality (3.1) is an equality, and tracing back, we obtain that

$$\begin{aligned}
\nabla_\Sigma w &= 0,\\
|A|^2 - \tfrac{1}{2}H^2 &= 0,\\
-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g &= \Lambda,\\
w_N &= \tfrac{1}{4-\gamma}h \text{ in } \Sigma,\\
H_{\partial M} + \gamma\partial_X w &= -h\cos\bar{\theta}_j \text{ along } \partial\Sigma\cap F_j,\\
\alpha_j &= \bar{\alpha}_j \text{ at } x \in \partial\Sigma\cap E_j.
\end{aligned}$$

(The second, third and fourth together are implied by $W = 0$). It remains to show that $R_\Sigma = 0$ and $\kappa_{\partial\Sigma} = 0$. First, we see that $\mathcal{A}''(0) = 0$ by the above. Let

$$\begin{aligned}
&Q(\psi,\psi)\\
=&\tfrac{4}{4-\gamma}\int_\Sigma |\nabla_\Sigma\psi|^2 + \left(\tfrac{1}{2}\int_\Sigma R_\Sigma\psi^2 + \sum_j\int_{\partial\Sigma\cap F_j}\kappa_{\partial\Sigma}\psi^2\right)\\
&-\left(\int_\Sigma W\psi^2 + \sum_j\int_{\partial\Sigma\cap F_j}\tfrac{1}{\sin\bar{\theta}_j}(H_{\partial M} + \gamma\partial_X w - \mu\cos\bar{\theta}_j)\psi^2\right).
\end{aligned}$$

Note that $Q(\psi,\psi)$ differs from the form (2.8) of $\mathcal{A}''(0)$ by only one term, and $Q(\psi,\psi) \geqslant \mathcal{A}''(0) = 0$. In fact,

$$Q(\psi,\psi) = \tfrac{4}{4-\gamma}\int_\Sigma |\nabla_\Sigma\psi|^2 + \left(\tfrac{1}{2}\int_\Sigma R_\Sigma\psi^2 + \sum_j\int_{\partial\Sigma\cap F_j}\kappa_{\partial\Sigma}\psi^2\right) \geqslant 0.$$

Let $\mathcal{L} = -\frac{4}{4-\gamma}\Delta_\Sigma + \frac{1}{2}R_\Sigma$, $\mathcal{B} = \frac{4}{4-\gamma}\partial_\nu + \kappa_{\partial\Sigma}$. By (3.2), $Q(1,1) = 0$, so $\mathcal{L}1 = 0$ and $\mathcal{B}1 = 0$, which gives $R_\Sigma = 0$ and $\kappa_{\partial\Sigma} = 0$. □

3.2. **Local foliation.** Now we construct a local foliation near an infinitesimally rigid Σ. We state here a slightly more general version.

Lemma 3.2. *Let Σ be a capillary surface of prescribed weighted mean curvature , if the linearization of $H+\gamma u^{-1}u_N+h$ is $-\Delta_\Sigma$ and the linearization of $\cos\theta-\cos\bar{\theta}$ is $-\sin\theta\frac{\partial}{\partial\nu}$, then there exists a local foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ near Σ such that $\Sigma_0=\Sigma$, $H+\gamma u^{-1}u_N+h$ is constant along Σ_t and $\theta=\bar{\theta}$ along $\partial\Sigma_t$.*

Remark 3.3. *A combination of* [CS25, Lemma 3.4] *and* [Li20a, Proposition 4.1] *finishes the proof, see also* [Ye91], [Amb15].

Now we derive an ODE for the quantity $\tilde{H}=H+\gamma u^{-1}u_N+h$ along the foliation. This step is the boundary version of [CS25, Lemma 4.4]. However, [CS25, Lemma 4.4] is only for dimension greater than three, here we make use of the Gauss-Bonnet theorem with boundary and turning angles.

Lemma 3.4. *Let $\{\Sigma_t\}$ be constructed in Lemma 3.2, assume that (Ω,g) satisfies the assumptions of Theorem 1.3, then*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\exp(-\int_0^t\Psi(s)\mathrm{d}s)\tilde{H})'\leqslant 0,$$

where

$$\Psi(t)=(\int_{\Sigma_t}\phi_t^{-1})^{-1}\left(-\tfrac{1}{2}\int_{\Sigma_t}(3\mu-\gamma w_N)+\sum_j\int_{\partial\Sigma_t\cap F_j}\cot\theta_j\right).$$

Proof. The first variation (2.6) gives

$$\phi_t^{-1}\tilde{H}'(t)=-\phi_t^{-1}\Delta_{\Sigma_t}\phi_t-\gamma u^{-1}\Delta_{\Sigma_t}u-\gamma u^{-1}\phi_t^{-1}\langle\nabla_{\Sigma_t}u,\nabla_{\Sigma_t}\phi_t\rangle+Z,$$

which is equivalent to the following

$$\phi_t^{-1}\tilde{H}'(t)=-\operatorname{div}_{\Sigma_t}\left(\tfrac{\nabla_{\Sigma_t}\phi_t}{\phi_t}+\gamma\nabla_{\Sigma_t}w\right)-(1-\tfrac{\gamma}{4})\phi_t^{-2}|\nabla_{\Sigma_t}\phi_t|^2-\gamma\left|\nabla_{\Sigma_t}w+\tfrac{\nabla_{\Sigma_t}\phi_t}{2\phi_t}\right|^2+Z.$$

Here, ϕ_t is the variational vector field of the foliation $\{\Sigma_t\}$. We integrate the above on Σ_t and using the divergence theorem,

$$
\begin{aligned}
&\tilde{H}' \int_{\Sigma_t} \phi_t^{-1} + \int_{\Sigma_t} \left((1 - \tfrac{\gamma}{4}) \phi_t^{-2} |\nabla_{\Sigma_t} \phi_t|^2 + \gamma \left| \nabla_{\Sigma_t} w + \tfrac{\nabla_{\Sigma_t} \phi_t}{2\phi_t} \right|^2 \right) \\
= & - \sum_j \int_{\partial \Sigma_t \cap F_j} (\phi_t \partial_{\nu_t} \phi_t + \gamma \partial_\nu w) + \int_{\Sigma_t} Z \\
= & - \int_{\partial \Sigma_t \cap F_j} (q_t + \gamma \partial_{\nu_t} w) + \int_{\Sigma_t} Z \\
= & \sum_j \int_{\partial \Sigma_t \cap F_j} (\kappa_{\partial \Sigma_t} - \tfrac{1}{\sin \bar{\theta}_j} (H_{\partial M} + \gamma \partial_X w - \mu \cos \theta_j)) + \tilde{H} \sum_j \int_{\partial \Sigma_t \cap F_j} \cot \theta_j \\
& + \int_{\Sigma_t} \left(-W + \tfrac{1}{2} R_\Sigma - \tfrac{3}{4} \tilde{H}^2 - \tfrac{1}{2} \tilde{H} (3\mu - \gamma w_N) \right),
\end{aligned}
$$

where we have used (2.12) and (2.13). Since $W \geqslant 0$,

$$
\tilde{H}' \int_{\Sigma_t} \phi_t^{-1} \leqslant \tilde{H} \left(-\tfrac{1}{2} \int_{\Sigma_t} (3\mu - \gamma w_N) + \sum_j \int_{\partial \Sigma_t \cap F_j} \cot \theta_j \right). \tag{3.3}
$$

Solving this ODE, we finish the proof. □

3.3. **Proof of rigidity of frustum.** Now we are ready to prove the dihedral rigidity conjecture for frusta.

Proof of Theorem 1.10 for frusta. Using the existence and regularity theory in [Li20a, Theorem 2.1], there exists a minimiser E to (2.1) such that $\Sigma = \operatorname{int} M \cap \partial E$ is $C^{1,\alpha}$ up to the corner (Li's theorem is based on scaling argument, and u will play no role in the limit.).

Given any Σ, we can define $\mathcal{A}$ as in (2.2). Let $F(t) = \mathcal{A}(\Sigma_t)$ where Σ_t is the leaf of the foliation in Lemma 3.2. Then by the first variation (2.3),

$$
F'(t) = \int_{\Sigma_t} (H + \gamma u^{-1} u_N + h) \mathrm{d}\mathcal{H}^{n-2},
$$

note that there is no boundary term because that the contact angle is $\theta_j = \bar{\theta}_j$ along the edges. Using Lemma 3.4, $F'(t) \leqslant 0$ for $t \geqslant 0$ and $F'(t) \geqslant 0$ for $t \leqslant 0$ which means that every Σ_t also gives rise to a minimiser to the functional. By Lemma 3.1, every Σ_t is infinitesimal rigid. Now we calculate the metric of (Ω, g) and u using the infinitesimal rigidity. Moreover, by tracing back the equality, we have that ϕ_t is constant. Using $w_N = -h/(4-\gamma)$ and $H = -\gamma w_N - h$, we see $H = 2(\gamma - 2)h/(4-\gamma) = -2\alpha$ which is constant. We now show that $Y^\perp$ is conformal. First, $\nabla_{\partial_i} N = H\partial_i$. Since $\langle Y, N\rangle$ is constant,

$$\begin{aligned} 0 &= \nabla_{\partial_i}\langle Y, N\rangle \\ &= \langle \nabla_{\partial_i} Y, N\rangle + \langle Y, \nabla_{\partial_i} N\rangle \\ &= Y\langle \partial_i, N\rangle - \langle \nabla_Y N, \partial_i\rangle + H\langle Y, \partial_i\rangle \\ &= -\langle \nabla_Y N, \partial_i\rangle + H\langle Y, \partial_i\rangle. \end{aligned}$$

Observe that

$$\nabla_Y N = \nabla_{Y^\top} N + \nabla_{Y^\perp} N = HY^\top + \nabla_{Y^\perp} N,\ \langle Y, \partial_i\rangle = \langle Y^\top, \partial_i\rangle,$$

hence $\langle \nabla_{Y^\perp} N, \partial_i\rangle = 0$. Moreover, $\nabla_{Y^\perp}\langle \partial_i, \partial_j\rangle = \langle Y^\perp, N\rangle g_{ij} = \frac{1}{2}H\phi_t g_{ij}$ by the umbilicity $|A|^2 - \frac{1}{2}H^2 = 0$. Note that every leaf is flat, therefore, the local foliation forms a subset $\cup_t \Sigma_t$ of the hyperbolic 3-space with constant curvature $-|\alpha|$. It follows from $w_N = -h/(4-\gamma)$ that $u = t^{-\beta/\alpha}$ (up to a constant). We now calculate the second fundamental form of $\partial\Omega$. Let e be a unit tangent vector of $\partial\Sigma$. Along the face F_j, using the decomposition $X_j = \cos\theta_j N + \sin\theta_j \nu$,

$$A_{F_j}(e, e) = \langle \nabla_e X, e\rangle = \cos\theta_j A(e, e) + \sin\theta_j \kappa_{\partial\Sigma} = \tfrac{1}{2}H\cos\theta_j.$$

It follows from $H_{\partial M} + \gamma\partial_X w = -h\cos\bar{\theta}_j$ that $H_{\partial\Omega} = H\cos\bar{\theta}_j$, and hence

$$A_{F_j}(\eta, \eta) = \langle \nabla_\eta X, \eta\rangle = H_{\partial\Omega} - \langle \nabla_e X, e\rangle = \tfrac{1}{2}H\cos\theta_j.$$

Note that the vector $N - \langle \eta, N\rangle\eta$ is of length $\sin\theta_j$, and the direction is the same with X, so

$$A_{F_j}(e, \eta) = \langle \nabla_e X, \eta\rangle = \tfrac{1}{\sin\theta_j}\langle \nabla_e(N - \langle \eta, N\rangle\eta), \eta\rangle = 0.$$

Hence, every face F_i is umbilic with curvature $\frac{1}{2}H\cos\theta_j$. In the upper half-space model of the hyperbolic 3-space, the face is either a part of a sphere or a plane. That each face F_i intersects the leaf Σ_t in a constant angle indicates that it can only be a part of a plane. Therefore, by connectedness, we can extend the rigidity to all Ω and (Ω, g) is a polyhedron in the upper half-space model. □

4. Rigidity of pyramids

In this section, we give the proof for rigidity of pyramids. Our method is to construct a local foliation near the apex which serves as a barrier for the existence of the minimisers to the warped μ-bubble functional.

First, we construct a local foliation near the apex O.

Proposition 4.1. *Let (Ω, g) and (Ω, δ) have isometric tangent cones at O. Then there exists a neighborhood U of O foliated by a family of surfaces $\{\Sigma_t\}_{t\in(-\varepsilon,0)}$ such that each Σ_t is of constant $H + \gamma\omega_N + h$ and meets the face F_i at the constant angle θ_i.*

Remark 4.2. *From now on, in some situations, we omit the dependence of θ and $\bar{\theta}$ on the indices of the faces for brevity.*

It is more useful to reformulate. The pyramid (Ω, δ) is formed by truncating its tangent cone at the apex through the base. We let Σ_1 be the cross-section parallel to the base and of unit distance to the apex O and Ω_1 to be the pyramid truncated by Σ_1. Let $\Sigma_t = t\Sigma_1$ and $\Omega_t = t\Omega_1$, $t > 0$. Let $x \in \Omega_1$, we define $v(x) = u(tx)$ and $(\hat{g}^t)_{ij}(x) = g_{ij}(tx)$.

We consider

$$\Sigma_{t,\phi} := \{(\check{x}, -t + \phi(\check{x})) : \ (\check{x}, -1) \in \Sigma_1\},$$

Let every geometric quantity of Σ_1 be denoted with a hat and a subscript t with respect to the metric $\hat{g} := \hat{g}^t$, and let every geometric quantity on $\Sigma_{1,\phi}$ be denoted by a hat and a subscript t, ϕ. For example, the unit normal of Σ_1 in Ω_1 with respect to the metric $\hat{g}$ is given by $\hat{N}_t$, and the unit normal of $\Sigma_{1,\phi}$ is given by $\hat{N}_{t,\phi}$.

By rescaling back using Proposition 4.3, we obtain the proof of Proposition 4.1. Indeed, let

$$\hat{\lambda}_{t,\phi} := \hat{H}_{t,\phi} + \gamma v^{-1} \partial_{\hat{N}_{t,\phi}} v + th.$$

Proposition 4.3. *There exists a family of functions $\{\phi(\cdot, t)\}_{t\in[0,\varepsilon)}$ defined on Σ_1 such that the perturbations $\Sigma_{1,t\phi(\cdot,t)}$ has constant $\lambda_{t,t\phi(\cdot,t)}$ and have constant angles with F_i with respect to the metric g^t.*

Proof. Let s be a small parameter and the family $\Sigma_{1,s\phi}$ give rise to a vector field $\partial_s := (0, \phi(\check{x}))$. The perturbation $\Sigma_{1,s\phi}$ of Σ_1 is approximately a normal graph over Σ_1 with the graph function $\hat{\phi}$ satisfying

$$\hat{\phi} := s\hat{g}(\partial_s, \hat{N}_t) + O(s^2),$$

Setting $s = t$, then the graph function

$$\hat{\phi} = t\phi + O(t^2)$$

since $\hat{g}$ converges to the flat metric.

By the first variation of $\lambda_{t,s\phi}$ and the Taylor expansion (with respect to s),

$$\lambda_{t,s\phi} - \lambda_{t,0} = \hat{L}_s\hat{\phi} + s\langle(\partial_s)^\top, \nabla^{\hat{g}_t}\lambda_{t,0}\rangle + O(s^2) = s\hat{L}_s\phi + s\langle(\partial_s)^\top, \nabla^{\hat{g}_t}\lambda_{t,0}\rangle O(s^2)$$

by (4) where $\hat{L}_s$ is define for Σ_1 as (2.6) with respect to the metric g^t. Setting $s = t$ yields

$$\lambda_{t,s\phi} = \lambda_{t,0} + t\hat{L}_t\phi + t\langle(\partial_s)^\top, \nabla^{\hat{g}_t}\lambda_{t,0}\rangle + O(t^2).$$

By convergence of $\hat{g}$ to the flat metric and u converges to the flat metric, $\hat{L}_t = -\Delta_{\Sigma_1} + O(t)$ where Δ_{Σ_1} is the Laplace-Beltrami operator with respect to the flat metric (i.e., limit of $\hat{g}$) on Σ_1, and $\langle(\partial_s)^\top, \nabla^{\hat{g}_t}\lambda_{t,0}\rangle = O(t)$. So

$$\lambda_{t,t\phi} = \lambda_{t,0} - \Delta_{\Sigma_1}\phi + O(t^2), \tag{4.1}$$

Similarly using (2.7),

$$\cos\hat{\theta}_{t,t\phi} = \cos\hat{\theta}_{t,0} - t\sin\hat{\theta}_{t,0}\tfrac{\partial\phi}{\partial\nu_t} + t\hat{q}_t\phi + t\langle(\partial_s)^{\partial\Sigma_1}, \nabla\hat{\theta}_{t,0}\rangle + O(t^2), \tag{4.2}$$

where $\hat{q}_t$ is defined in (2.4) for Σ_1 with respect to the metric g^t.

Since the (Ω_1, g^t) converges to the flat pyramid and u converges to a constant, $\hat{L}_t = -\Delta_{\Sigma_1} + O(t)$ and similarly, $\sin\hat{\theta}\frac{\partial}{\partial\nu_t} = \sin\theta\frac{\partial}{\partial\nu_1} + O(t)$ and $q_t = O(t)$.

Define

$$\Psi(t,\phi) = \left(\tfrac{1}{t}\lambda_{t,t\phi} - \tfrac{1}{|\Sigma_1|}\int_{\Sigma_1}\tfrac{1}{t}\lambda_{t,t\phi}, \frac{1}{t\sin\theta}(\cos\hat{\theta}_{t,t\phi} - \cos\bar{\theta})\right),$$

which can be extended to $t = 0$ by taking limits $\Psi(0,\phi) = \lim_{t\to 0}\Psi(t,u)$. By the expansion (4.1) and (4.2),

$$\Psi(0,\phi) = (-\Delta_{\Sigma_1}\phi + \tfrac{1}{|\Sigma_1|}\int_{\Sigma_1}\Delta_{\Sigma_1}\phi, -\tfrac{\partial\phi}{\partial\nu_1} + \zeta),$$

where $\zeta := \lim_{t\to 0}\frac{\cos\hat{\theta}_{t,0} - \cos\bar{\theta}}{t\sin\theta}$ is a function on $\partial\Sigma_1$.

By minimising the functional

$$I(\phi) = \int_{\Sigma_1}|\nabla_{\Sigma_1}\phi|^2 + \int_{\Sigma_1}\zeta\phi$$

on the space

$$\Lambda_0 = \{\phi \in C^{2,\alpha}(\Sigma_1)\cap C^{1,\alpha}(\bar{\Sigma}_1) : \int_{\Sigma_1}\phi = 0\},$$

we can find a solution to the elliptic problem $\Psi(0,\phi) = 0$, and we set the solution to be ϕ_0.

Now we compute

$$\begin{aligned} D\Psi|_{(0,\phi_0)}(0,\phi) =& \tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\Psi(0,\phi_0 + s\phi) \\ =& \tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=0}s(-\Delta_{\Sigma_1}\phi + \tfrac{1}{|\Sigma_1|}\int_{\Sigma_1}\Delta_{\Sigma_1}\phi, -\tfrac{\partial\phi}{\partial\nu}) \\ =& (-\Delta_{\Sigma_1}\phi + \tfrac{1}{|\Sigma_1|}\int_{\Sigma_1}\Delta_{\Sigma_1}\phi, -\tfrac{\partial\phi}{\partial\nu}), \end{aligned}$$

since ϕ_0 satisfies $\Psi(0,\phi_0) = 0$. Now we apply the implicit function theorem. For some sufficiently small $\varepsilon > 0$, there exists a function $\phi(\cdot,t) \in B(0,\delta) \subset \mathcal{X}$, $t \in (0,\varepsilon)$ such that $\phi(\cdot,0) = \phi_0$ and

$$\Psi(t,\phi(\cdot,t)) = \Psi(0,\phi_0) = (0,0)$$

for every $t \in [0, \varepsilon)$. In geometric terms, the surface $\Sigma_{t,\phi(\cdot,t)}$ are of constant $\lambda_{t,\phi(\cdot,t)}$ with constant contact angles $\bar{\theta}_j$ with the face F_j. □

Lemma 4.4. *Let $\Sigma_{1,t\phi(\cdot,t)}$ be constructed as in Proposition 4.3, then*

$$\lambda_{t,t\phi}|\Sigma_1| = \int_{\Sigma_1} \lambda_{t,0} + \int_{\partial\Sigma_1} \tfrac{1}{\sin\theta}(\cos\bar{\theta} - \cos\theta) + O(t^2).$$

Proof. This follows from (4.1), (4.2) by integration over Σ_1 and an application of the divergence theorem. □

We give a variational formula which gives a relation of the variations of $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g$, $H + \gamma u^{-1}\partial_N u$ and the dihedral angles.

Proposition 4.5. *Let $\{u_t\}$ be a family of positive C^2 functions and $\{g_t\}$ be a family of smooth metrics on Ω_1 converging respectively to the constant 1 and the flat metric as $t \to 0$. Then*

$$\begin{aligned}
&[-\int_{\Sigma_1} (H_g + \gamma u^{-1}\partial_N u) + \int_{\partial\Sigma} \tfrac{1}{\sin\theta}(\cos\bar{\theta} - \cos\theta)] \\
={}& \int_{\Omega_1} (-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g) + \int_{\partial\Omega_1\backslash\Sigma_1} (H_g + \gamma u^{-1}\partial_X u) + O(t^2).
\end{aligned}$$

Proof. The case u_t is a constant for all t is due to [MP22], so we only have to show

$$-\int_{\partial\Omega_1} u^{-1}\partial_N u = \int_{\Omega_1} u^{-1}\Delta_g u + \int_{\partial\Omega_1\backslash\Sigma_1} u^{-1}\partial_{X_i} u + O(t^2).$$

The above follows from that $u^{-1} = 1 + O(t)$ and the divergence theorem. □

By taking the difference between (u_1, g_1) and (u_2, g_2), we obtain the following.

Corollary 4.6. *Let $\{u_t^{(i)}\}_{i=1,2}$ be two families of positive C^2 functions and $\{g_t^{(i)}\}_{i=1,2}$ be two families of smooth metrics on Ω_1 converging respectively to the constant 1 and the flat metric as $t \to 0$. Then*

$$[-\int_{\Sigma_1} ((H_{g_1} + \gamma u^{-1}\partial_N u) - (H_{g_2} + \gamma u^{-1}\partial_N u)) + \int_{\partial\Omega_1} \tfrac{1}{\sin\theta}(\cos\theta^{(1)} - \cos\theta^{(2)})]$$

$$\begin{aligned}
={}& \int_{\Omega_1} ((-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g) - (-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g)) \\
&+ \int_{\partial\Omega_1\backslash\Sigma_1} ((H_g + \gamma u^{-1}\partial_X u) - (H_g + \gamma u^{-1}\partial_X u)) + O(t^2).
\end{aligned}$$

Now we are ready to finish the proof of Theorem 1.3 for pyramids.

Proof of Theorem 1.10 for pyramids. Note that

$$\lambda_{t,t\phi} = H_{1,t\phi} + \gamma u^{-1}\partial_N u + th,$$

where th is $-(H_{1,t\phi}+\gamma u^{-1}\partial_N u)$ computed with respect to the model. Hence, $\lambda_{t,t\phi}$ is the difference of the weighted mean curvatures with respect to two different metrics. Note that $\bar{\theta}$ is the same with the model. Hence subsequent applications of Lemma 4.4 and Corollary 4.6 shows that $\lambda_{t,t\phi} \geqslant O(t^2)$. Equivalently, by rescaling back, we obtain that $\tilde{H}_t \geqslant O(t)$ for the foliation $\{\Sigma_t\}$ in Proposition 4.1. The condition $\tilde{H}_t \geqslant O(t)$ gives an initial value for the ordinary differential inequality (3.3) (with a reversed direction) which is easily seen to hold for $\{\Sigma_t\}$ as well. We write the ODE here

$$\tilde{H}' \geqslant \tilde{H}\Psi(t), \Psi(t) = (\tfrac{1}{2}\int_{\Sigma_t}(3h+\gamma w_N) - \sum_j \int_{\partial\Sigma_t\cap F_j} \cot\theta_j)(\int_{\Sigma_t}\phi_t^{-1})^{-1}$$

We see that $\phi_t = 1+O(t)$, $\sum_j \int_{\partial\Sigma_t\cap F_j}\cot\theta_j = Ct+O(t^2)$ for some constant $C>0$ (see Remark 4.7) since the foliation in Proposition 4.1 is constructed from higher order perturbations of coordinate bases. Hence $\Psi(t) = Ct^{-1} + C_1(t)$ where $C_1(t)$ is a bounded continuous function. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{H}t^C \exp(\int^t C_1(s)\mathrm{d}s)) \geqslant 0$$

and we obtain that $\tilde{H} \geqslant 0$ for every leaf. This gives a barrier for the existence of minimiser to the capillary warped μ-functional for the polyhedron $\Omega\backslash E_t$ obtained by chopping off the pyramid over Σ_t. Applying the proof for the rigidity of frustums, we know that the rigidity holds $\Omega\backslash E_t$, which by taking a limit $t\to 0$, we obtain the rigidity for the pyramids. □

Remark 4.7. *We explain why $\sum_j \int_{\partial\Sigma_t\cap F_j}\cot\theta_j = Ct+O(t^2)$ holds. It suffices to consider the flat metric and $t=1$ by rescaling. Let O' be the projection of O to the plane where Σ_1 lies and $E_j = F_j\cap\partial\Sigma_1$. We see then that $\cot\theta_j$ is the signed distance of O' to the line where E_j lies formed by E_j. So $|\partial\Sigma_1\cap F_j|\cot\theta_j$ is twice the area of an oriented triangle formed by O' and E_j, and summing over all F_j gives twice the area of Σ_1, that is, the constant C.*

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