# A TILTED SPACETIME POSITIVE MASS THEOREM

XIAOXIANG CHAI

Abstract. We develop a mass type invariant asymptotically flat initial data sets with a non-compact boundary. We show a corresponding spacetime positive mass theorem for spin initial data sets under the dominant energy condition and a suitable dominant energy condition on the boundary which we call the tilted dominant energy condition.

## 1. Introduction

The positive mass theorem states that if a complete manifold which is asymptotically flat and with non-negative scalar curvature, an quantity called the ADM mass defined at infinity is non-negative. It was proved by Schoen and Yau in their seminal work [SY79] using a minimal surface technique. The ADM mass is a characterization of scalar curvature at infinity. There are various works on the positive mass theorem: [Wit81], [EHLS16], [ACG08], [Wan01], [CH03], [Sak21]. Here the list is by no means exhaustive.

The study of the positive mass type theorems of the asymptotically flat manifold with a non-compact boundary was initiated in the work [ABdL16]. The effect of the mean curvature was included to the infinity and a boundary term was added to the ADM mass. See [AdL20], [AdLM19], [Cha18], [Cha21] and [AdL22] for some developments to the spacetime and the hyperbolic settings.

Almaraz-de Lima-Mari [AdLM19] introduced the asymptotically flat initial data sets and prove a version of the spacetime positive mass theorem. In this paper, we revisit the asymptotically flat initial data set with a non-compact boundary. In particular, we introduce a new boundary dominant condition (1.2) and prove the spacetime positive mass theorem (Theorems 1.4) for spin initial data sets.

### 1.1. Asymptotically flat initial data sets with a non-compact boundary.

An *initial data set* $(M^n, g, p)$ is an $n$-dimensional manifold which arises as a spacelike hypersurface of a Lorentzian manifold $(\mathcal{S}^{n,1}, \tilde{g})$ with $p$ being the second fundamental form. The components $T_{00}$ and $T_{0i}$ of the Einstein tensor (or the energy-momentum tensor) $T$ are respectively called the *energy density* $\mu$ and the *current density* $J$. Let $e_0$ be the future directed unit normal of $M$ to $\mathcal{S}$, $e_i$ be an orthonormal basis of the tangent space of $M$ and we use the convention on $p$ that $p_{ij} = \tilde{g}(\tilde{\nabla}_{e_i} e_0, e_j)$.

The energy density $\mu$ by the Gauss equation is

$$2\mu = R_g + (\operatorname{tr}_g p)^2 - |p|_g^2$$

2020 *Mathematics Subject Classification.* 53C50, 83C05.

*Key words and phrases.* Tilted dominant energy condition, spacetime positive mass theorem, spinor, mass, linear momentum, asymptotically flat initial data set.

and the current density $J$ by the Gauss-Codazzi equation is

$$J = \operatorname{div} p - g\mathrm{d}(\operatorname{tr}_g p).$$

**Definition 1.1.** *We say that $(M, g, p)$ satisfies the interior dominant energy condition if*

$$\mu \geqslant |J|. \tag{1.1}$$

*If $\partial M \neq \emptyset$, let $\eta$ be the outward normal of $\partial M$ in $M$, $H_{\partial M} = \operatorname{div}_{\partial M} \eta$. We say that $(M, g, p)$ satisfies the tilted boundary dominant energy condition on the boundary $\partial M$ if*

$$H_{\partial M} + \cos\theta \operatorname{tr}_{\partial M} p \geqslant \sin\theta |p(\eta, \cdot)^\top| \text{ on } \partial M, \tag{1.2}$$

*where $\theta \in [0, \pi]$ is a constant angle parameter and $p(\eta, \cdot)^\top$ denotes the component of the 1-form $p(\eta, \cdot)$ tangential to $\partial M$.*

The tilted boundary dominant energy condition (1.2) generalizes the tangential ($\theta = \frac{\pi}{2}$) and normal boundary dominant energy conditions ($\theta = 0$) in [AdLM19]. Now we recall the definition of an asymptotically flat initial data set with a non-compact boundary and its ADM energy and linear momentum of [AdLM19].

**Definition 1.2.** *We say that an initial data set $(M, g, p)$ is asymptotically flat with a non-compact boundary if there exists a compact set $K$ such that $M$ is diffeomorphic to $\mathbb{R}^n_+ \backslash B$ (the Euclidean half-space minus a ball) and*

$$|g - \delta| + |x||\partial g| + |x|^2|\partial^2 g| + |x||p| + |x|^2|\partial p| = o(r^{-\frac{n-2}{2}}), \tag{1.3}$$

*where $B$ is the standard Euclidean ball of a fixed radius.*

**Definition 1.3.** *Assume that $\mu + |J| \in L^1(M)$ and $H_{\partial M} + |p(\eta, \cdot)^\top| \in L^1(\partial M)$, then the quantities defined as*

$$E = \lim_{r\to\infty} \left[ \int_{S_+^{n-1,r}} (g_{ij,j} - g_{jj,i})\nu^i - \int_{S^{n-2,r}} e_{\alpha n}\vartheta^\alpha \right],$$

*and*

$$P_i = 2\int_{S_+^{n-1,r}} \pi_{ij}\nu^j.$$

*are finite and are respectively called the ADM energy and ADM linear momentum. Here, $\nu$ is unit normal to $S_+^{n-1,r}$, $\vartheta$ is normal to $S^{n-2,r}$ in $\partial M$ and $\pi = p - g\operatorname{tr}_g p$. Denote $\hat{P} = (P_1, \ldots, P_{n-1})$, $S_+^{n-1,r}$ is the upper half of the coordinate sphere of radius $r$ and $S^{n-2,r} = \partial S_+^{n-1,r}$.*

Note that we have included $P_n$ in the ADM linear-momentum as well, and this is a key difference from [AdLM19, Definition 2.4].

1.2. **Tilted positive mass theorem.** We use the spinorial argument of Witten [Wit81] (see also [PT82]). We have the following two spacetime positive mass theorems.

**Theorem 1.4.** *If $(M, g)$ is spin and $(M, g, p)$ satisfies the interior dominant energy condition* (1.1) *and the tilted boundary dominant energy condition* (1.2) *for some nonzero $\theta \in [0, \pi]$, then*

$$E + \cos\theta P_n \geqslant \sin\theta |\hat{P}|. \tag{1.4}$$

The special case $\theta = \frac{\pi}{2}$ of the theorem is due to [AdLM19]. As we shall see later, (1.4) is related to an energy-momentum vector (2.1). Throughout this paper, we assume that $\theta \in (0, \pi)$, and the case $\theta = 0$, $\pi$ only requires minor modifications.

The time-symmetric case $p = 0$ of the theorem first appeared in [ABdL16] where a minimal surface proof was also given. Here is the rough idea in dimension 3. Assume that the energy (mass) $E$ is negative, we can perturb the metric so that it is harmonically flat at infinity, the scalar curvature and the mean curvature of the boundary are strictly positive. Then the boundary $\partial M$ and a plane asymptotically parallel to $\partial M$ serve as the barriers and we can find an area-minimizing minimal surface which is asymptotic to a coordinate plane that lies in between. Then the Gauss-Bonnet theorem applied on the stable minimal plane contradicts the strict positivity of the scalar curvature and the mean curvature. An alternative proof was given by the author [Cha18], where the free boundary minimal surface was used instead. Assume that $E < 0$, we can construct a free boundary area-minimizing surface that lies in between two coordinate half-planes. The existence of such a free boundary minimal surface again contradicts the Gauss-Bonnet theorem.

Observing the two works and [EHLS16], the two proofs are actually for the two special cases when $p$ vanishes: (I) $\theta = \pi/2$ in [Cha18]; (II) or $\theta = 0$, $\pi$ in [ABdL16]. It is then reasonable to expect a proof of the more general Theorem 1.4 using the capillary marginally outer trapped surface, see [ALY20]. While it is a possible approach to Theorem 1.4, the construction of capillary MOTS remains a technical problem.

**Definition 1.5.** *Let $\Sigma$ be a hypersurface in the initial data set $(M, g, k)$, the quantity $\theta^+ = H_\Sigma + \operatorname{tr}_\Sigma p$ ($\theta^- = H_\Sigma - \operatorname{tr}_\Sigma p$) is called outer (inner) null expansion. If $\theta^+ = 0$ ($\theta^- = 0$) along $\Sigma$, then $\Sigma$ is called a marginally outer (inner) trapped hypersurface, in short MOTS (MITS). If $\Sigma \cap \partial M$ is nonempty, $\Sigma$ and $\partial M$ forms a constant contact angle $\theta$, then we say $\Sigma$ is a capillary MOTS.*

We also need the capillary MOTS in the rigidity case of Theorem 1.4.

1.3. **Rigidity.** The rigidity of the positive mass theorem was recently studied in many works, for example, [HL23, HL20, HL23, HL24, HZ24]. It was found that initial data sets of zero mass do not necessarily isometrically embeds into the Minkowski time, see [BC96, HL24]. We call an initial data set with zero mass a *rigid initial data sets*. In general, the rigid initial data sets lies within the so-called plane-fronted waves with parallel propagation or in short pp-wave spacetime.

**Definition 1.6.** *We say a manifold $\mathcal{S}^{n,1}$ with a metric $\tilde{g}$ of Lorentzian signature is a pp-wave spacetime if $\mathcal{S}^{n,1} = \mathbb{R}^{n+1}$ and*

$$\tilde{g} = -2\mathrm{d}u\mathrm{d}t + F\mathrm{d}u^2 + \delta_{\mathbb{R}^{n-1}},$$

*where $F$ is independent of $t$ and superharmonic on $\mathbb{R}^{n-1} \times \{u\}$ for all $u \in \mathbb{R}$.*

A rough version of the equality case of Theorem 1.4 is given in the following. For a precise statement, see Theorem 5.21.

**Theorem 1.7.** *If the equality*

$$E + \cos\theta P_n = \sin\theta |\hat{P}|$$

*is achieved in (1.4) of Theorem 1.4, then $(M, g, p)$ is foliated by flat capillary MOTS. Moreover, it admits an isometric embedding into a pp-wave spacetime with the* *second fundamental form p, in particular,* $(M, g, p)$ *isometrically embeds into the half Minkowski spacetime if* $E + \cos\theta P_n = 0$*.*

This theorem can be seen as a boundary analog of Hirsch-Zhang [HZ24]. Our approach combines an observation similar to [HZ24] that the equality in the Witten's spinor proof [Wit81] is achieved by a set of spinors and a paper of the author with Wan [CW24] on the dihedral rigidity of initial data sets.

We use the spinor spacetime spinor bundle in the proof, however, it is not necessary in even dimensions. Because the Clifford multiplication by the timelike unit vector $e_0$ can be replaced by the Clifford multiplication of the complex volume element in the modified connection (3.1) on the spacetime spinor bundle and the chirality operator (3.3) when the dimension is even. See [AdL22] where the operator (3.3) was originally introduced. It is an interesting question to find a proof without using the spacetime spinor bundle in odd dimensions. This is possible in the usual case of spacetime positive mass theorem, the case by [HZ24], since there is no boundary involved. We just replace the connection in (3.1) by

$$\nabla_i + \tfrac{1}{2}(-1)^\sigma \sqrt{-1} p_{ij} e_j \cdot,$$

and proceed similarly using the techniques in Section 5.

The article is organized as follows:

In Section 2, we describe the mass related Theorems 1.4 and show the invariance. In Section 3, we collect basics of the chirality operator (3.3) and the hypersurface Dirac operator including the most important Schrodinger-Lichnerowicz formula. In Section 4, we give the proofs of Theorems 1.4. In Section 5, we give the proofs of the rigidity statement Theorem 1.7.

**Acknowledgments**

I would like to thank Xueyuan Wan (CQUT, Chongqing), Tin Yau Tsang (UCI) and Martin Li (CUHK) for various discussions, and Levi Lima (UFC, Brazil) for communicating the preprint [AdL22]. Part of this work was carried out when I was a member of Korea Institute for Advanced Study (KIAS) under the grant No. MG074402. I was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. RS-2024-00337418) and an NRF grant No. 2022R1C1C1013511.

## 2. The invariance of mass

In this section, we introduce the energy-momentum vector $(E^\theta, P_i^\theta)$ in (2.1) based on the Hamiltonian analysis in [HH96] and point out that the tilted dominant energy condition (1.2) appears in selecting a suitable lapse function and the shift vector.

2.1. **Hamiltonian formulation and mass invariance.** Assume at present that $M$ is compact, we infinitesimally deform the initial data set $(M, g, p)$ in $\mathcal{S}^{n,1}$ in the direction of a future directed timelike vector field $T$. Let $\phi_s$ be the local flow of $T$, $M_s = \phi_s(M)$. We assume that the unit normal $e_0$ to $M_s$ is always tangential to the timelike hypersurface foliated by $\partial M_s$. Let $T = Ne_0 + X$, where $N$ is called the *lapse* function and the vector field $X$ tangent to $M$ is called the *shift vector*, then the Hamiltonian along $M$ is given by (see [HH96])

$$\mathcal{H}(N,X)=\int_M[N\mu+2J(X)]+2\int_{\partial M}[NH-p(X,\eta)+\operatorname{tr}_g p\langle X,\eta\rangle].$$

The tilted boundary dominant energy condition (1.2) now comes from selecting $N=1$ and $X=\cos\theta\eta+\sin\theta\tau$ where $\tau$ is tangent to $\partial M$ in the boundary term of the Hamiltonian. Indeed,

$$\begin{aligned}&NH-p(X,\eta)+\operatorname{tr}_g p\langle X,\eta\rangle\\ =&H-\cos\theta p(\eta,\eta)-\sin\theta p(\tau,\eta)+\cos\theta\operatorname{tr}_g p\\ =&H+\cos\theta\operatorname{tr}_{\partial M}p-\sin\theta p(\tau,\eta),\end{aligned}$$

which is non-negative if (1.2) holds.

Now let $(M,g,p)$ be the background $(\mathbb{R}^n_+,\delta,0)$, we take $N$ to be a constant and $X$ be a translational Killing vector field of $\mathbb{R}^n_+$. We consider the Hamiltonian $\mathcal{H}_\varepsilon(N,X)$ on $(M,\delta+\varepsilon g,\varepsilon p)$ with $(g,p)$ satisfying (1.3). We do the Taylor expansion of $\mathcal{H}_\varepsilon$ with respect to $\varepsilon$, due to the fact that $M$ is non-compact, usually the first order terms do not vanish. These terms evaluated at infinity are precisely those given in Definition 1.3. For a more complete account of these facts, we refer the readers to [HH96], [Mic11] and [AdLM19].

We define the *charge density* $\mathbb{U}$ which is a 1-form,

$$\begin{aligned}&\mathbb{U}_{(g,k)}(N,X)\\ =&N(\operatorname{div}_\delta g-\mathrm{d}(\operatorname{tr}_\delta g))-(g-\delta)(\nabla^\delta N,\cdot)\\ &+\operatorname{tr}_\delta(g-\delta)\mathrm{d}N+2(p(X,\cdot)-\operatorname{tr}_\delta p\langle\cdot,X\rangle_\delta).\end{aligned}$$

Let $\mathcal{T}$ be the space of translational Killing vector fields of Minkowski spacetime denoted by $\mathbb{R}^{1,n}$. It is easy to see that $\mathcal{T}$ is identified with $\mathbb{R}\oplus W$ with $\mathbb{R}$ factor representing the translation in a chosen timelike direction $\partial_0$ and $W$ being the linear space spanned by all translational Killing vector fields of $(\mathbb{R}^n,\delta)$ orthogonal to $\partial_0$. Each $T\in\mathcal{T}$ can be uniquely written in the form $T=N\partial_0+X^i\partial_i$ where $N\in\mathbb{R}$ and $X^i\in\mathbb{R}$. We define the energy-momentum functional as follows:

$$\mathcal{M}(T)=\lim_{r\to\infty}\left[\int_{S^{n-1,r}_+}\mathbb{U}_{(g,k)}(N,X)+\int_{S^{n-2,r}}Ng(\bar\eta,\bar\vartheta)\right].$$

It was shown in [AdLM19, Proposition 3.3] that the energy-momentum functional $\mathcal{M}(T)$ does not depend on the asymptotic coordinates (fixing $\partial_0$) chosen at infinity.

For any $\theta\in(0,\pi)$, we define

$$E^\theta=\mathcal{M}(\tfrac{1}{\sin\theta}\partial_0+\tfrac{\cos\theta}{\sin\theta}\partial_n),P^\theta_i=\mathcal{M}(\partial_i)\text{ for any }i\neq n. \tag{2.1}$$

It is easy to check that $E=\mathcal{M}(\partial_0)$, $P_i=\mathcal{M}(\partial_i)$ where $(E,P)$ is as defined in Definition 1.3, so $E^\theta=\frac{1}{\sin\theta}E+\frac{\cos\theta}{\sin\theta}P_n$. We have the following.

**Theorem 2.1.** *Given any asymptotically flat initial data set $(M,g,k)$, for any $\theta\in(0,\pi)$, the vector $(E^\theta,P^\theta)\in\mathbb{R}^{1,n-1}$ is well defined (up to composition with an element of $SO_{1,n-1}$). In particular,*

$$-(E^\theta)^2+\sum_{i\neq n}(P^\theta_i)^2$$

*and the causal character $(E^\theta,P^\theta)\in\mathbb{R}^{1,n-1}$ do not depend on the chart at infinity to compute $(E^\theta,P^\theta)$.*

*Proof.* Let $\tilde{\partial}_0 = \frac{1}{\sin\theta}(\partial_0 + \cos\theta\partial_n)$, $\tilde{\partial}_n = \frac{1}{\sin\theta}(\cos\theta\partial_0 + \partial_n)$ and $\tilde{\partial}_i = \partial_i$. There is a Lorentz boost from $(\partial_0, \partial_1, \dots, \partial_n)$ to $(\tilde{\partial}_0, \tilde{\partial}_1, \dots, \tilde{\partial}_n)$ such that

$$\begin{pmatrix} \tilde{\partial}_0 \\ \tilde{\partial}_n \end{pmatrix} = \begin{pmatrix} \cosh\rho & \sinh\rho \\ \sinh\rho & \cosh\rho \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_n \end{pmatrix},$$

on the plane spanned by $\{\partial_0, \partial_n\}$ with $\rho$ defined by $\cosh\rho = \frac{1}{\sin\theta}$. So $(\tilde{\partial}_0, \tilde{\partial}_1, \dots, \tilde{\partial}_n)$ gives a new coordinate system for the Minkowski spacetime $\mathbb{R}^{1,n}$. Let $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n) \in \mathbb{R}^{1,n}$ where $\tilde{x}$ is expressed in the new coordinates. Obviously,

$$-(\tilde{x}_0)^2 + \sum_{i\neq n} (\tilde{x}_n)^2$$

is invariant under linear Lorentz transformations of $\mathbb{R}^{1,n}$ which fixes $\tilde{\partial}_n$. These transformations as a subgroup of the special Lorentz group $SO_{1,n}$ is isomorphic to $SO_{1,n-1}$. The discussion applies to

$$(\mathcal{M}(\tilde{\partial}_0), \mathcal{M}(\widetilde{\partial_1}), \dots, \mathcal{M}(\tilde{\partial}_{n-1}), \mathcal{M}(\tilde{\partial}_n)),$$

and this is our theorem. □

For the cases $\theta = 0$, $\pi$, it is simpler.

**Theorem 2.2.** *Given any asymptotically flat initial data set $(M, g, k)$, the quantity $E \pm P_n$ is a numerical invariant under isometries of $\mathbb{R}^n_+$ which includes rotations and translations of the $(n-1)$-dimensional hyperplane $\partial\mathbb{R}^n_+$.*

*Proof.* Note that $E$ and $P_n$ are invariant under rotations and translations of the hyperplane $\{\partial_1, \dots, \partial_{n-1}\}$, see [AdLM19, Proposition 3.3]. □

## 3. Hypersurface Dirac operator

In this section, we recall the hypersurface Dirac spinors and the related Schrodinger-Lichnerowicz formula (3.2). We review the chirality operator (3.3) and we relate the boundary condition (3.4) to the geometric quantities along the boundary $\partial M$ in Lemma 3.5.

3.1. **Hypersurface Dirac operator.** The standard reference of spin geometry is [LM89], we also refer to [PT82], [HZ03]. Denote by $\mathbb{S}$ the local spinor bundle of $\mathcal{S}^{n,1}$, since $M$ is spin, $\mathbb{S}$ exists globally over $M$. This spinor bundle $\mathbb{S}$ is called the hypersurface spinor bundle of $M$. When our spacetime $\mathcal{S}$ is of dimension $3+1$, the local spinor bundle $\mathbb{S}$ have a simpler algebraic description by the representation theory of the special linear group $\mathrm{SL}(2, \mathbb{C})$. In this case, the theory is easier to understand, see [PT82, Section 2].

Let $\tilde{\nabla}$ and $\nabla$ denote respectively the Levi-Civita connections of $\tilde{g}$ and $g$, we use the same symbols to denote the lifts of the connections to the hypersurface spinor bundle. There exists a Hermitian inner product $(\cdot, \cdot)$ on $\mathbb{S}$ over $M$ which is compatible with the spin connection $\tilde{\nabla}$. For any vector $e$ of $\mathcal{S}$ and the hypersurface spinors $\phi$, $\psi$, we have

$$(e \cdot \phi, \psi) = (\phi, e \cdot \psi)$$

where the dot $\cdot$ denotes the Clifford multiplication. This inner product is not positive definite. However, there exists on $\mathbb{S}$ over $M$ a positive definite Hermitian inner product defined by

$$\langle \phi, \psi \rangle = (e_0 \cdot \phi, \psi)$$

where $e_0$ is the future-directed unit timelike normal to $M$. We see that

$$\langle e_0 \cdot \phi, \psi \rangle = \langle \phi, e_0 \cdot \psi \rangle, \ \langle e_i \cdot \phi, \psi \rangle = -\langle \phi, e_i \cdot \psi \rangle,$$

where $\{e_i\}$ is an orthonormal basis over $M$. Then the spinor connection $\tilde{\nabla}$ over $\mathbb{S}$ is related to $\nabla$ by

$$\tilde{\nabla}_i = \nabla_i - \tfrac{1}{2} p_{ij} e_j \cdot e_0 \cdot .$$

This is essentially the spinorial Gauss equation. Moreover, the connection $\nabla$ is compatible with $\langle \cdot, \cdot \rangle$ and $\nabla_i (e_0 \cdot \phi) = e_0 \cdot \nabla_i \phi$.

For our purpose, we extend $\tilde{\nabla}$ to $\tilde{\nabla}^{(\sigma)}$ defined by

$$\tilde{\nabla}^{(\sigma)}_i \phi = \nabla_i \phi + \tfrac{1}{2}(-1)^\sigma p_{ij} e_j \cdot e_0 \cdot \phi, \tag{3.1}$$

where $\sigma$ is an integer. The *hypersurface Dirac (or Dirac-Witten) operator* is then given by

$$\tilde{D}^{(\sigma)} = e_i \cdot \tilde{\nabla}^{(\sigma)}_i = D - \tfrac{1}{2}(-1)^\sigma \operatorname{tr}_g p e_0 \cdot,$$

where $D$ is the standard Dirac operator. We also call a spinor $\phi$ satisfying $\tilde{D}\phi = 0$ a *(spacetime) harmonic spinor.*

From here after, $\tilde{\nabla}$ and $\tilde{D}$ will be referring to $\tilde{\nabla}^{(\sigma)}$ and $\tilde{D}^{(\sigma)}$. When there is a possible confusion, we will indicate the supscripts explicitly.

The integration form of the Schrodinger-Lichnerowicz formula (see [PT82]) is given as follows.

**Theorem 3.1.** *Let $\Omega$ be a compact manifold with boundary, we have for any smooth spinor $\phi$ that*

$$\begin{aligned} &\int_\Omega |\tilde{D}\phi|^2 - \int_\Omega |\tilde{\nabla}\phi|^2 + \int_{\partial\Omega} [\langle \nu \cdot \tilde{D}\phi, \phi \rangle + \langle \phi, \tilde{\nabla}_\nu \phi \rangle] \\ =& \tfrac{1}{2} \int_\Omega \langle (\mu + (-1)^\sigma J \cdot e_0 \cdot)\phi, \phi \rangle, \end{aligned} \tag{3.2}$$

*where $\nu$ is the outward unit normal of $\partial\Omega$.*

3.2. **Boundary chirality operator.** We fix the conventions first. We use the Greek letters $\alpha$, $\beta$, $\gamma$ to indicate the indices which are not $n$ in the rest of the paper. The vector $e_n$ is used to denote the outer normal of $\partial M$ in $M$ and $h$ denotes the the second fundamental form of $\partial M$ given by $h_{\alpha\beta} = \langle \nabla_{e_\alpha} e_n, e_\beta \rangle$, then $H_{\partial M} = \sum_\alpha h_{\alpha\alpha}$.

The following chirality operator was introduced by [AdL22, Definition 3.3] where $e_0$ is replaced by the Clifford multiplication of the complex volume element.

**Definition 3.2.** *Given an integer $\sigma_1$, define $Q^{(\sigma_1)}$ by*

$$Q^{(\sigma_1)}\phi = \cos\theta e_n \cdot e_0 \cdot \phi + (-1)^{\sigma_1} \sqrt{-1} \sin\theta e_n \cdot \phi. \tag{3.3}$$

*When there is no confusion, we write $Q = Q^{(\sigma_1)}$ and we also use the convention $Q \cdot \phi = Q\phi$.*

We collect the commutative and anti-commutative properties of $Q$ below.

**Lemma 3.3.** *The operator $Q$ satisfies the following*

*(a) $Q \circ Q = 1$ and $Q$ is self-adjoint;*
*(b) $e_n \cdot Q + Q \cdot e_n = -2(-1)^{\sigma_1} \sqrt{-1} \sin\theta$;*
*(c) $e_\alpha \cdot e_\beta \cdot e_n \cdot Q + Q \cdot e_\alpha \cdot e_\beta \cdot e_n \cdot = -(-1)^{\sigma_1} 2\sqrt{-1} \sin\theta e_\alpha \cdot e_\beta \cdot$;*

(d) $e_0 \cdot Q + Q \cdot e_0 = 0$;
(e) $e_\alpha \cdot e_\beta \cdot e_0 \cdot Q + Qe_\alpha \cdot e_\beta \cdot e_0 \cdot = 0$;
(f) $e_\alpha \cdot Q\phi - Q \cdot e_\alpha = 2(-1)^{\sigma_1}\sqrt{-1}\sin\theta e_\alpha \cdot e_n$;
(g) $e_\alpha \cdot Q\phi + Q \cdot e_\alpha = 2\cos\theta e_\alpha \cdot e_n \cdot e_0$;
(h) $e_\alpha \cdot e_n \cdot Q + Q \cdot e_\alpha \cdot e_n = 0$;
(i) $e_n \cdot e_0 \cdot Q + Q \cdot e_n \cdot e_0 = 2\cos\theta$;
(j) $e_\alpha \cdot e_\beta \cdot e_n \cdot e_0 \cdot Q + Q \cdot e_\alpha \cdot e_\beta \cdot e_n \cdot e_0 = 2\cos\theta e_\alpha \cdot e_\beta$;
(k) $e_\alpha \cdot e_0 \cdot Q + Qe_\alpha \cdot e_0 = 2(-1)^{\sigma_1}\sqrt{-1}\sin\theta e_\alpha \cdot e_0 \cdot e_n$;
(l) for $\alpha \neq \beta$, $e_\alpha \cdot e_\beta \cdot Q = Qe_\alpha \cdot e_\beta \cdot$;
(m) $e_\alpha \cdot e_n \cdot e_0 \cdot Q = Q \cdot e_\alpha \cdot e_n \cdot e_0$.

*Proof.* All the items follows from direct calculation starting from the definition of $Q$ in (3.3). As an example, we only show the last item. By (3.3) and anti-commutative property of the Clifford multiplication,

$$\begin{aligned} e_\alpha \cdot e_n \cdot e_0 \cdot Q &= \cos\theta e_\alpha \cdot e_n \cdot e_0 \cdot e_n \cdot e_0 + (-1)^{\sigma_1}\sqrt{-1}\sin\theta e_\alpha \cdot e_n \cdot e_0 \cdot e_n \\ &= \cos\theta e_\alpha + (-1)^{\sigma_1}\sqrt{-1}\sin\theta e_\alpha \cdot e_0, \\ Q \cdot e_\alpha \cdot e_n \cdot e_0 \cdot &= \cos\theta e_0 \cdot e_n \cdot e_\alpha \cdot e_n \cdot e_0 + (-1)^{\sigma_1}\sqrt{-1}\sin\theta e_n \cdot e_\alpha \cdot e_n \cdot e_0 \\ &= \cos\theta e_\alpha + (-1)^{\sigma_1}\sqrt{-1}\sin\theta e_\alpha \cdot e_0. \end{aligned}$$

So the last item holds. □

3.3. **Boundary terms in Schrodinger-Lichnerowicz formula.** We calculate the term $\langle \nu \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_\nu \phi\rangle$ along $\partial M$ when

$$Q\phi = (-1)^{\sigma_2}\phi, \tag{3.4}$$

where $\sigma_2$ is an integer. First, we compute a few inner products of spinors satisfying (3.4).

**Lemma 3.4.** *If a spinor $\phi$ satisfies* (3.4) *along $\partial M$, then*

$$\langle \sqrt{-1}e_n \cdot \phi, \phi\rangle = (-1)^{\sigma_1+\sigma_2}\sin\theta|\phi|^2, \tag{3.5}$$

$$\langle e_n \cdot e_0 \cdot \phi, \phi\rangle = (-1)^{\sigma_2}\cos\theta|\phi|^2, \tag{3.6}$$

$$\langle e_\alpha \cdot e_0 \cdot \phi, \phi\rangle = (-1)^{\sigma_1+\sigma_2}\sin\theta\langle\sqrt{-1}e_\alpha \cdot e_0 \cdot e_n \cdot \phi, \phi\rangle. \tag{3.7}$$

*Proof.* The first term (3.5) already appeared in [AdL22, Proposition 3.11]. From Lemma 3.3, we have

$$\langle \sqrt{-1}e_n \cdot Q\phi, \phi\rangle + \langle Q \cdot \sqrt{-1}e_n \cdot \phi, \phi\rangle = 2(-1)^{\sigma_1}\sin\theta|\phi|^2.$$

Since $Q$ is self-adjoint, so

$$\langle \sqrt{-1}e_n \cdot Q\phi, \phi\rangle + \langle \sqrt{-1}e_n \cdot \phi, Q\phi\rangle = 2(-1)^{\sigma_1}\sin\theta|\phi|^2.$$

Because $Q\phi = (-1)^{\sigma_2}\phi$, we have

$$2\langle \sqrt{-1}e_n \cdot \phi, \phi\rangle = 2(-1)^{\sigma_1+\sigma_2}\sin\theta|\phi|^2,$$

which is the first item. The rest follow similarly from corresponding relations from Lemma 3.3. □

The following lemma relates the boundary term in the integration form of Schrodinger-Lichnerowicz formula (3.2) with the mean curvature $H$, $\mathrm{tr}_{\partial M} p$, $p_{nj}$ along the boundary, and in particular, the tilted boundary dominant energy condition (1.2).

**Lemma 3.5.** *If a spinor $\phi$ satisfies* (3.4) *along $\partial M$, then*

$$\begin{aligned}&\langle\tilde{\nabla}_{e_n}\phi+e_n\cdot\tilde{D}\phi,\phi\rangle\\=&\langle D^{\partial M}\phi,\phi\rangle-\tfrac{1}{2}H_{\partial M}|\phi|^2-\tfrac{1}{2}(-1)^{\sigma+\sigma_2}\cos\theta\operatorname{tr}_{\partial M}p|\phi|^2\\&+\tfrac{1}{2}(-1)^{\sigma+\sigma_1+\sigma_2}\sin\theta\langle\sqrt{-1}p_{n\gamma}e_\gamma\cdot e_0\cdot e_n\cdot\phi,\phi\rangle.\end{aligned}$$

*Proof.* Let $D^{\partial M}$ be the boundary Dirac operatoror defined by

$$D^{\partial M}=e_n\cdot e_\alpha\cdot\nabla_\alpha^{\partial M}.$$

Here, $\nabla^{\partial M}$ is the spin connection intrinsic to $\partial M$ explicitly defined on spinor fields on $M$ restricted to $\partial M$ as

$$\nabla_\alpha^{\partial M}=\nabla_\alpha-\tfrac{1}{2}h_{\alpha\beta}e_n\cdot e_\beta\cdot.$$

We calculate $D^{\partial M}\phi$ with $\phi$ satisfying (3.4) and

$$\begin{aligned}&D^{\partial M}\phi\\=&e_n\cdot e_\alpha\cdot(\nabla_\alpha\phi-\tfrac{1}{2}h_{\alpha\beta}e_n\cdot e_\beta\cdot\phi)\\=&e_n\cdot(D\phi-e_n\cdot\nabla_{e_n}\phi)+\tfrac{1}{2}H_{\partial M}\phi\\=&e_n\cdot D\phi+\nabla_{e_n}\phi+\tfrac{1}{2}H_{\partial M}\phi\\=&e_n\cdot\left(\tilde{D}\phi+\tfrac{1}{2}(-1)^\sigma\operatorname{tr}_g pe_0\cdot\phi\right)+\left(\tilde{\nabla}_{e_n}\phi-\tfrac{1}{2}(-1)^\sigma p_{nj}e_j\cdot e_0\cdot\phi\right)+\tfrac{1}{2}H_{\partial M}\phi\end{aligned}$$

So

$$\begin{aligned}&\langle\tilde{\nabla}_{e_n}\phi+e_n\cdot\tilde{D}\phi,\phi\rangle\\=&\langle D^{\partial M}\phi,\phi\rangle-\tfrac{1}{2}(-1)^\sigma\langle\operatorname{tr}_g pe_n\cdot e_0\cdot\phi-p_{nj}e_j\cdot e_0\cdot\phi,\phi\rangle-\tfrac{1}{2}H_{\partial M}|\phi|^2.\end{aligned}$$

It remains to calculate

$$\begin{aligned}&\langle\operatorname{tr}_g pe_n\cdot e_0\cdot\phi-p_{nj}e_j\cdot e_0\cdot\phi,\phi\rangle\\=&\langle(\operatorname{tr}_{\partial M}pe_n+p_{nn}e_n)\cdot e_0\cdot\phi-(p_{n\alpha}e_\alpha+p_{nn}e_n)\cdot e_0\cdot\phi,\phi\rangle\\=&\operatorname{tr}_{\partial M}p\langle e_n\cdot e_0\cdot\phi,\phi\rangle-\langle p_{n\alpha}e_\alpha\cdot e_0\cdot\phi,\phi\rangle\\=&(-1)^{\sigma_2}\cos\theta\operatorname{tr}_{\partial M}p|\phi|^2+(-1)^{\sigma_1+\sigma_2+1}\langle p_{n\alpha}e_\alpha\cdot e_0\cdot\phi,\phi\rangle\\=&(-1)^{\sigma_2}\cos\theta\operatorname{tr}_{\partial M}p|\phi|^2+(-1)^{\sigma_1+\sigma_2+1}\sin\theta\langle\sqrt{-1}p_{n\alpha}e_\alpha\cdot e_0\cdot e_n\cdot\phi,\phi\rangle,\end{aligned}$$

which follows from (3.6) and (3.7). □

## 4. THE POSITIVE MASS THEOREM

In this section, we prove the tilted spacetime positive mass theorem (Theorems 1.4).

4.1. **Existence of a spacetime harmonic spinor.** When the initial data set $(M,g,p)$ is flat and totally geodesic, i.e. $(M,g,p)$ is $(\mathbb{R}^n_+,\delta,0)$, we define

$$\bar{Q}\phi:=\bar{Q}^{(\sigma_1)}\phi=\cos\theta\tfrac{\partial}{\partial x^n}\cdot\tfrac{\partial}{\partial x^0}\cdot\phi+(-1)^{\sigma_1}\sqrt{-1}\sin\theta\tfrac{\partial}{\partial x^n}\cdot\phi.$$

Note that $\bar{Q}^2=I$, and the eigenvalues of $\bar{Q}$ are $\pm1$. The standard hypersurface spinor bundle $\bar{\mathbb{S}}$ over $(\mathbb{R}^n_+,\delta,0)$ splits into two eigen subbundles and the spinor $\phi$ satisfying

$$\bar{Q}\phi=(-1)^{\sigma_2}\phi\tag{4.1}$$

is closely related to our problem. Here $\sigma_1$ and $\sigma_2$ are the two integers defined earlier in (3.3) and (3.4).

We recall the following existence of a spacetime harmonic spinor $\phi$ which is asymptotic to a constant spinor $\phi_0$ satisfying (4.1), and we extract the mass from the boundary integral in (3.2). By [AdLM19, Proposition 5.3] and the discussions that followed, we have the following.

**Theorem 4.1.** *Assume that $(M, g, k)$ satisfies the dominant energy conditions (1.1) and (1.2), and let $\sigma = \sigma_2$. Given any nonzero constant spinor $\phi_0$ satisfying (4.1), there exists a spinor $\phi$ which is asymptotic to $\phi_0$ and satisfies*

$$\begin{aligned} \tilde{D}\phi &= 0 \text{ in } M, \\ Q\phi &= (-1)^\sigma \phi \text{ on } \partial M. \end{aligned}$$

4.2. **Proof of positive mass theorems.** Using the $\phi$ of Theorem 4.1 in (3.2), we can give a proof of Theorem 1.2.

*Proof of Theorem 1.4.* Let $M_r$ be the compact region bounded by $\partial M$ and $S_r^{n-1,+}$. By the integral form of Schrodinger-Lichnerowicz formula (3.2), we have for any spinor $\phi$, we have

$$\begin{aligned} & \int_{M_r} |\tilde{D}\phi|^2 - \int_{M_r} |\tilde{\nabla}\phi|^2 + \int_{\partial M_r} [\langle e_i \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_i \phi\rangle] * e_i \\ = & \tfrac{1}{2} \int_{M_r} \langle (\mu + (-1)^\sigma J \cdot e_0 \cdot)\phi, \phi\rangle. \end{aligned}$$

Note that $\partial M_r$ are made of two portions: one lies in the interior of $M$ and the other lies on $\partial M$. Note that $Q\phi = (-1)^\sigma \phi$ along $\partial M$, so by Lemma 3.5,

$$\begin{aligned} & \int_{M_r} |\tilde{D}\phi|^2 - \int_{M_r} |\tilde{\nabla}\phi|^2 + \int_{\partial M_r \cap \operatorname{int} M} [\langle \nu \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_\nu \phi\rangle] \\ & + \int_{\partial M_r \cap \partial M} \left[\langle D^{\partial M}\phi, \phi\rangle - \tfrac{1}{2} H|\phi|^2 - \tfrac{1}{2}\cos\theta \operatorname{tr}_{\partial M} p|\phi|^2\right] \\ & \quad + \tfrac{1}{2}(-1)^{\sigma_1} \int_{\partial M_r \cap \partial M} \sin\theta \langle \sqrt{-1} p_{n\gamma} e_\gamma \cdot e_n \cdot e_0 \cdot \phi, \phi\rangle \\ = & \tfrac{1}{2} \int_{M_r} \langle (\mu + (-1)^\sigma J \cdot e_0 \cdot)\phi, \phi\rangle. \end{aligned}$$

It follows that $\langle D^{\partial M}\phi, \phi\rangle = 0$ from [CH03, (4.27)] (with $\varepsilon$ there replaced by $Q$). We claim that

$$\begin{aligned} & \int_{\partial M_r \cap \operatorname{int} M} [\langle \nu \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_\nu \phi\rangle] \\ \to & \tfrac{1}{4}(E + \cos\theta P_n)|\phi_0|_\delta^2 + \sin\theta (-1)^{\sigma_1} P_\gamma \langle \sqrt{-1} \tfrac{\partial}{\partial x^\gamma} \cdot \tfrac{\partial}{\partial x^n} \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\rangle_\delta \end{aligned}$$

as $r \to \infty$ to finish the proof. Indeed, we proceed by calculation. First,

$$\begin{aligned} & \langle \nu \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_\nu \phi\rangle \\ = & \langle \nu \cdot D\phi, \phi\rangle + \langle \phi, \nabla_\nu \phi\rangle \\ & + \tfrac{1}{2}(-1)^\sigma (p_{ij}\nu^j - \operatorname{tr}_g p \nu^i)\langle e_i \cdot e_0 \cdot \phi, \phi\rangle \\ = & \langle \nu \cdot D\phi, \phi\rangle + \langle \phi, \nabla_\nu \phi\rangle + \tfrac{1}{2}(-1)^\sigma \pi_{ij}\nu^j \langle e_i \cdot e_0 \cdot \phi, \phi\rangle, \end{aligned}$$

where $\nu$ is the unit normal of $\partial M_r \cap \operatorname{int} M$ pointing to the infinity. Because that $\phi$ converges to a constant spinor $\phi_0$ and $\langle e_i \cdot e_0 \cdot \phi, \phi\rangle$ converges to the constant $\langle \frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\rangle_\delta$, so as $r \to \infty$

$$\int_{\partial M_r \cap \operatorname{int} M} \langle \nu \cdot D\phi, \phi\rangle + \langle \phi, \nabla_\nu \phi\rangle \to \tfrac{1}{4} E|\phi_0|^2_\delta$$

from [ABdL16, Section 5.2] and

$$\tfrac{1}{2}\int_{\partial M_r \cap \operatorname{int} M} \pi_{ij}\nu^j \langle e_i \cdot e_0 \cdot \phi, \phi\rangle \to \tfrac{1}{4} P_i \left\langle \tfrac{\partial}{\partial x^i} \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\right\rangle_\delta .$$

Here $\delta$ is the standard Euclidean metric. It follows from (3.6) that

$$P_n \left\langle \tfrac{\partial}{\partial x^n} \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\right\rangle_\delta = (-1)^\sigma P_n \cos\theta |\phi_0|^2_\delta,$$

and from (3.7) that

$$\sin\theta P_\gamma \left\langle \tfrac{\partial}{\partial x^\gamma} \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\right\rangle_\delta = (-1)^{\sigma_1+\sigma} \sin\theta P_n \left\langle \sqrt{-1}\tfrac{\partial}{\partial x^\gamma} \cdot \tfrac{\partial}{\partial x^n} \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\right\rangle_\delta .$$

Up to here, we finish the proof of the claim. Now, given the above considerations, as $r \to \infty$,

$$\begin{aligned} &\tfrac{1}{4}(E + \cos\theta P_n)|\phi_0|^2_\delta + \sin\theta(-1)^{\sigma_1} P_\gamma \left\langle \sqrt{-1}\tfrac{\partial}{\partial x^\gamma} \cdot \tfrac{\partial}{\partial x^n} \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\right\rangle_\delta \\ &= \int_M |\tilde\nabla \phi|^2 + \tfrac{1}{2}\int_M \langle (\mu + (-1)^\sigma J \cdot e_0 \cdot)\phi, \phi\rangle \\ &+ \tfrac{1}{2}\int_{\partial M} \left[(H + \cos\theta \operatorname{tr}_{\partial M} p)|\phi|^2 + \sin\theta(-1)^{\sigma_1}\left\langle \sqrt{-1} p_{n\gamma} e_\gamma \cdot e_n \cdot e_0 \cdot \phi, \phi\right\rangle\right] . \end{aligned} \tag{4.2}$$

Let

$$A = P_\gamma \sqrt{-1}\tfrac{\partial}{\partial x^\gamma} \cdot \tfrac{\partial}{\partial x^n} \cdot \tfrac{\partial}{\partial x^0},$$

we know from the last item of Lemma 3.3 that $\bar Q$ commutes with $A$ and they have the same eigen-spinors. It is not difficult to see that eigenvalues of $A$ are $\pm|\hat P|$. For a fixed $\sigma_1$, we make a choice of $\phi_0$ and $\sigma$ such that

$$P_\gamma \left\langle \sqrt{-1}\tfrac{\partial}{\partial x^\gamma} \cdot \tfrac{\partial}{\partial x^n} \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_0, \phi_0\right\rangle = (-1)^{\sigma_1+1}|\hat P||\phi_0|^2, \tag{4.3}$$

which by the dominant energy conditions (1.1) and (1.2), leads immediately to the mass inequality $E + \cos\theta P_n \geqslant \sin\theta|\hat P|$. □

## 5. Analysis of the rigidity

Our main assumption of this section is vanishing mass, that is,

$$E + \cos\theta P_n = \sin\theta|\hat P|. \tag{5.1}$$

5.1. **Analysis of spinors.** Let $\phi_0$ be a spinor satisfying (4.3) and $\phi$ is given in Theorem 4.1. Assume in addition (5.1) holds, then it is easy to see from (4.2) that

$$\tilde\nabla^{(\sigma)}_i \phi = \nabla_i \phi + \tfrac{1}{2}(-1)^\sigma p_{ij} e_j \cdot e_0 \cdot \phi = 0 \tag{5.2}$$

And $\phi$ is also subject to the boundary condition

$$Q^{(\sigma)}\phi = \kappa e_n \cdot e_0 \cdot \phi + (-1)^{\sigma_1}\tau\sqrt{-1} e_n \cdot \phi = (-1)^\sigma \phi \text{ along } \partial M. \tag{5.3}$$

Hereafter, we set $\kappa = \cos\theta$ and $\tau = \sin\theta$ for convenience. We fix $x_1$ direction so that $\frac{\partial}{\partial x^1} = \hat P|\hat P|^{-1}$ if $\hat P \neq 0$. If $\hat P = 0$, then we take any direction orthogonal to $x_n$ direction to be $x_1$.

We are going to use the following lemma quite often.

**Lemma 5.1.** *Let $\phi$ be a non-zero spinor and $E_1$ be a unit vector with*

$$\langle\sqrt{-1}E_1\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle=\max_{|E|=1}\langle\sqrt{-1}E\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle.$$

*Then*

$$\langle\sqrt{-1}E_2\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle=0$$

*where $E_2$ is any vector orthogonal to $E_1$. If $\langle\sqrt{-1}E_1\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle=|\phi|^2$, then*

$$\sqrt{-1}E_1\cdot\partial_0\cdot\partial_n\cdot\phi=\phi.$$

*Proof.* It is always true that

$$\langle\sqrt{-1}E_1\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle\leqslant|\phi|^2.$$

And the equality is achieved if and only if $\sqrt{-1}E_1\cdot\partial_0\cdot\partial_n\cdot\phi=\phi$. Let $E(t)$, $t\in(-\varepsilon,\varepsilon)$ be a short smooth curve such that $|E(t)|=1$, $E(0)=E_1$ and $E'(0)=E_2$. The function $f(t):=\langle\sqrt{-1}E\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle$ attains its maximum at $t=0$ by the assumptions. Hence,

$$f'(0)=\langle\sqrt{-1}E'(0)\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle=\langle\sqrt{-1}E_2\cdot\partial_0\cdot\partial_n\cdot\phi,\phi\rangle=0,$$

which finishes the proof. □

*Remark* 5.2. Lemma 5.1 holds for other vector-related operators on spinors, and we will refer to Lemma 5.1 when it applies.

**Lemma 5.3.** *Assume that $\phi$ is a spinor which is asymptotic to the constant spinor $\phi_0$ of unit length which satisfies* (5.2) *and* (5.3)*. Assume in addition*

$$\sqrt{-1}\partial_1\cdot\partial_0\cdot\partial_n\cdot\phi_0=(-1)^{\sigma_1+1}\phi_0. \tag{5.4}$$

*Let $f=\langle\phi,\phi\rangle$ and $W_i=\langle e_i\cdot e_0\cdot\phi,\phi\rangle$, then $f=|W|$.*

*Proof.* By a direct calculation,

$$\nabla_i f=-(-1)^\sigma p_{ij}W_j,\ \nabla_i W_j=-(-1)^\sigma p_{ij}f. \tag{5.5}$$

It follows then $\nabla_i(f^2-|W|^2)=0$. Hence $f^2-|W|^2$ is a constant. We choose an orthonormal frame $\{e_i\}$ on the boundary $\partial M$ such that $e_1$ is asymptotic to $\frac{\partial}{\partial x^1}$, and calculate the components of $W$. First, we calculate $W_n$ along the boundary,

$$\begin{aligned}W_n&=\langle e_n\cdot e_0\cdot\phi,\phi\rangle\\&=\tfrac{1}{2}(-1)^\sigma\langle e_n\cdot e_0\cdot Q\cdot\phi,\phi\rangle+\tfrac{1}{2}(-1)^\sigma(e_n\cdot e_0\cdot\phi,Q\cdot\phi)\\&=\kappa(-1)^\sigma|\phi|^2.\end{aligned}$$

So $W_n$ tends to $\kappa(-1)^\sigma$ as $|x|\to\infty$. For $W_\alpha$,

$$\begin{aligned}W_\alpha=&\langle e_\alpha\cdot e_0\cdot\phi,\phi\rangle\\=&\tfrac{1}{2}(-1)^\sigma\langle e_\alpha\cdot e_0\cdot Q\cdot\phi,\phi\rangle+\tfrac{1}{2}(-1)^\sigma(e_\alpha\cdot e_0\cdot\phi,Q\cdot\phi)\\=&(-1)^{\sigma+\sigma_1}\tau\langle\sqrt{-1}e_\alpha\cdot e_0\cdot e_n\cdot\phi,\phi\rangle.\end{aligned}$$

By (5.4), $W_1$ tends to $(-1)^{\sigma+1}\tau$. For $\alpha\neq1$, then by Lemma 5.1 and (5.4),

$$\langle\sqrt{-1}\partial_\alpha\cdot\partial_0\cdot\partial_n\cdot\phi_0,\phi_0\rangle=0,\ \alpha\neq1.$$

So $W_\alpha\to0$ as $|x|\to\infty$. Hence $f^2-|W|^2$ limits to zero as $|x|\to\infty$. Therefore, $f=|W|$. □

*Remark* 5.4. Note that we cannot derive that $f^2 - |W|^2$ is asymptotic to zero without assuming the boundary condition (5.3). Consideration of $f^2 - |W|^2$ at the boundary is necessary.

**Lemma 5.5.** *Setting $\xi = W/f$, then*

$$\xi \cdot e_0 \cdot \phi = \phi, \ \langle e \cdot e_0 \cdot \phi, \phi \rangle = 0 \tag{5.6}$$

*where $e$ is orthogonal to $\xi$. And along $\partial M$,*

$$\langle \xi, e_n \rangle = (-1)^\sigma \cos\theta, \ (-1)^{\sigma+\sigma_1}\sqrt{-1}|\xi^\top|^{-1}\xi^\top \cdot e_0 \cdot e_n \cdot \phi = \phi,$$

*where $\xi^\top$ is the component of $\xi$ tangential to $\partial M$.*

*Proof.* It follows from the boundary condition $Q\phi = (-1)^\sigma \phi$, more specifically, (3.6) and (3.7) that

$$\begin{aligned} f &= \langle \xi \cdot e_0 \cdot \phi, \phi \rangle \\ &= \langle \xi, e_n \rangle \langle e_n \cdot e_0 \cdot \phi, \phi \rangle + \langle \xi^\top \cdot e_0 \cdot \phi, \phi \rangle \\ &= (-1)^\sigma \langle \xi, e_n \rangle \kappa |\phi|^2 + \tau |\xi^\top| \left\langle \sqrt{-1} \tfrac{(-1)^{\sigma+\sigma_1}\xi^\top}{|\xi^\top|} \cdot e_0 \cdot e_n \cdot \phi, \phi \right\rangle. \end{aligned}$$

Assume that $\langle \xi, e_n \rangle = (-1)^\sigma \cos\theta_1$ and $|\xi^\top| = \sin\theta_1$ where $\theta_1 \in [0,\pi]$ is a function on $\partial M$, then

$$\begin{aligned} f = |\phi|^2 =& \cos\theta_1 \cos\theta |\phi|^2 + \sin\theta \sin\theta_1 \left\langle \sqrt{-1}|\xi^\top|^{-1}(-1)^{\sigma+\sigma_1}\xi^\top \cdot e_0 \cdot e_n \cdot \phi, \phi \right\rangle \\ \leqslant& |\phi|^2 (\cos\theta_1 \cos\theta + \sin\theta \sin\theta_1) \\ =& |\phi|^2 \cos(\theta - \theta_1), \end{aligned}$$

which forces $\theta_1 = \theta$ along $\partial M$. In particular, it gives $\langle \xi, e_n \rangle = (-1)^\sigma \cos\theta$ and

$$\left\langle \sqrt{-1}|\xi^\top|^{-1}(-1)^{\sigma+\sigma_1}\xi^\top \cdot e_0 \cdot e_n \cdot \phi, \phi \right\rangle = |\phi|^2 \text{ along } \partial M. \tag{5.7}$$

The above gives the rest of the lemma. □

*Remark* 5.6. We see

$$(-1)^\sigma \xi^\top |\xi^\top|^{-1} \to -\partial_1, \ (-1)^\sigma \xi \to -\kappa \tfrac{\partial}{\partial x^n} - \tau \tfrac{\partial}{\partial x^1} \tag{5.8}$$

as $|x| \to \infty$.

5.2. **Multiple spinor components.** Let $\phi$ be a spinor given in Lemma 5.3, we fix the convention $\xi_\phi$ to denote the vector field given in Lemma 5.5.

**Lemma 5.7.** *Let $\psi$ and $\phi$ be two spinors given in Lemma 5.3 corresponds to $\sigma = \chi_1$ and $\sigma = \chi_2$ respectively where $\chi_1$ and $\chi_2$ are two integers. Then $(-1)^{\chi_2}\xi_\phi = (-1)^{\chi_1}\xi_\psi$.*

*Proof.* Set $z = \langle \psi, \psi \rangle - \langle \phi, \phi \rangle$ and $Z_j = \langle e_j \cdot e_0 \cdot \psi, \psi \rangle - (-1)^{\sigma_2 - \sigma_1} \langle e_j \cdot e_0 \cdot \phi, \phi \rangle$. Calculating

$$\nabla_i z = -(-1)^{\sigma_1} p_{ij} Z_j, \ \nabla_i Z_j = -(-1)^{\sigma_1} p_{ij} z$$

directly which gives $\nabla_i (z^2 - |Z|^2) = 0$, so $z^2 - |Z|^2$ is constant. It follows from (5.8) that $\xi_\psi \to (-1)^{\chi_2 - \chi_1}\xi_\phi$ as $x \to \infty$, and the only non-zero component of the limit of $Z$ is the limit of $\langle Z, \xi_\phi\rangle\xi_\phi$, which is actually zero. Moreover, we see easily $z \to 0$ as $|x| \to \infty$. Hence, $z^2 = |Z|^2$ which yields

$$\begin{aligned}&\langle\psi,\psi\rangle^2 + \langle\phi,\phi\rangle^2 - 2\langle\phi,\phi\rangle\langle\psi,\psi\rangle\\ =&\sum_i(\langle e_i\cdot e_0\cdot\psi,\psi\rangle^2 - 2(-1)^{\chi_2-\chi_1}\langle e_i\cdot e_0,\psi,\psi\rangle\langle e_i\cdot e_0\cdot\phi,\phi\rangle + \langle e_i\cdot e_0\cdot\phi,\phi\rangle^2).\end{aligned}$$

By Lemma 5.3,

$$\langle\phi,\phi\rangle\langle\psi,\psi\rangle = \sum_i(-1)^{\chi_2-\chi_1}\langle e_i\cdot e_0,\psi,\psi\rangle\langle e_i\cdot e_0\cdot\phi,\phi\rangle.$$

The right hand side reduces to the following

$$\langle\phi,\phi\rangle\langle\psi,\psi\rangle = (-1)^{\chi_2-\chi_1}\langle\xi_\phi\cdot e_0\cdot\psi,\psi\rangle\langle\xi_\phi\cdot e_0\cdot\phi,\phi\rangle = (-1)^{\chi_2-\chi_1}\langle\phi,\phi\rangle\langle\xi_\phi\cdot e_0\cdot\psi,\psi\rangle$$

by applying (5.6) for the spinor $\phi$, which gives $\langle\psi,\psi\rangle = (-1)^{\chi_2-\chi_1}\langle\xi_\phi\cdot e_0\cdot\psi,\psi\rangle$. So $\xi_\phi = (-1)^{\chi_2-\chi_1}\xi_\psi$ which proves the lemma. □

Now we show the orthogonality of spinor solutions to (5.2) everywhere if they are orthogonal at infinity.

**Proposition 5.8.** *Let $\{\phi_{0,i}\}_{i=1,2}$ be two spinor satisfying* (4.3) *with $\sigma_1$ replaced by two integers $\{\sigma_{1,i}\}$ and $\{\phi_i\}$ be given in Theorem 4.1 with suitably chosen $\sigma = \sigma_{(i)}$ corresponding to $\sigma_{1,i}$. Assume in addition* (5.1) *holds, and $\{\phi_{0,i}\}$ are of unit length and orthogonal, then*

$$|\phi_1| = |\phi_2|,\ \langle\phi_1,\phi_2\rangle = 0$$

*everywhere.*

*Proof.* Since $|\phi_i| > 0$, everywhere, we can assume that $|\phi_1| = c|\phi_2|$ at some point of $M$ for some positive constant. We set $\nu = (-1)^{\sigma_{(i)}}\xi_{\phi_i}$ by Lemma 5.7, and a simple calculation gives

$$\begin{aligned}\nabla_j(|\phi_1|^2 - c|\phi_2|^2) =& - p_{jk}((-1)^{\sigma_{(1)}}\langle e_k\cdot e_0\cdot\phi_1,\phi_1\rangle - (-1)^{\sigma_{(2)}}(e_k\cdot e_0\cdot\phi_2,\phi_2))\\ =& - (p(e_j,(-1)^{\sigma_{(1)}}\xi_{\phi_1})|\phi_1|^2 - p(e_j,(-1)^{\sigma_{(2)}}\xi_{\phi_2})|\phi_2|^2)\\ =& - p_{j\nu}(|\phi_1|^2 - c|\phi_2|^2).\end{aligned}$$

Then $|\phi_1|^2 = c|\phi_2|^2$ everywhere by an ODE argument. The norms of both $\phi_1$ and $\phi_2$ approaches to 1, so $c = 1$. It remains to show $\phi_1$ and $\phi_2$ are orthogonal. If $\sigma_{(1)} = \sigma_{(2)} + 1$, then

$$\nabla_j\langle\phi_1,\phi_2\rangle = 0$$

by (5.2). Combining with that $\langle\phi_1,\phi_2\rangle$ approaches zero as $|x| \to \infty$ leads to $\langle\phi_1,\phi_2\rangle = 0$. If $\sigma_{(1)} = \sigma_{(2)}$, we can assume that $\nu\cdot e_0\cdot\phi_i = \phi_i$. It is easy to check $\frac{1}{\sqrt{2}}(\phi_1+\phi_2)$ and $\frac{1}{\sqrt{2}}(\phi_1-\phi_2)$ satisfy the assumptions of the proposition, hence, they have the same norm. Similarly, for $\frac{1}{\sqrt{2}}(\phi_1+\sqrt{-1}\phi_2)$ and $\frac{1}{\sqrt{2}}(\phi_1-\sqrt{-1}\phi_2)$. The polarization yields $\langle\phi_1,\phi_2\rangle = 0$. □

5.3. **Foliation by flat capillary MOTS.** In this section, we carry an argument used by [BC96] for a single spinor component.

**Proposition 5.9.** *Let $\phi$ be a spinor, and $f$ and $W$ be the function and the vector field associated with it given in Lemma 5.3,*

*(a) there exists a global foliation of $M$ such that $W$ is normal to each leaf;*

(b) *each leaf* $\Sigma$ *is a capillary MOTS; in particular, the unit normal of* $\Sigma$ *is given by* $\nu = (-1)^\sigma \xi_\phi$ *and let* $h^\Sigma = \nabla \nu$ *be the second fundamental form of* $\Sigma$ *in* $M$*, then* $h^\Sigma + p_{|\Sigma} = 0$*;*

(c) *The spinor* $\phi|\phi|^{-1}$ *is parallel with respect to the induced spinor connection of* $\Sigma$*.*

*Proof.* Around a point $q$ near infinity, let $\{\varepsilon_i\}$ be an orthonormal frame such that $\nabla_{\varepsilon_i}\varepsilon_j = 0$ at $q$. Let $\{\varepsilon^i\}$ be the dual frame and $\omega$ be the dual 1-form of $W$. Here, the components of tensors or forms are taken with respect to the frame $\{e_i\}$. Then $\omega = W_i \varepsilon^i$ and the calculation

$$
\begin{aligned}
&\mathrm{d}\omega(\varepsilon_i, \varepsilon_j) \\
=&\varepsilon_i(\omega(\varepsilon_j)) - \varepsilon_j(\omega(\varepsilon_i)) \\
=&\varepsilon_i\langle W, \varepsilon_j\rangle - \varepsilon_j\langle W, \varepsilon_i\rangle \\
=&\langle \nabla_i W, \varepsilon_j\rangle - \langle \nabla_j W, \varepsilon_i\rangle \\
=&-(-1)^\sigma(\langle fq(\varepsilon_i), , \varepsilon_j\rangle - \langle fq(\varepsilon_j), \varepsilon_i\rangle) \\
=&0,
\end{aligned}
$$

shows that $\omega$ is a closed 1-form. By the Frobenius theorem, there exists a foliation of $M$ such that $W$ is normal to each leaf, say $\Sigma$.

Note that $\xi_\phi = W/f$ is a unit normal to $\Sigma$, let $\nu = (-1)^\sigma \xi_\phi$. We can assume that $\varepsilon_n = \nu$ and the indices $i, j < n$. Then by (5.5),

$$
\begin{aligned}
h^\Sigma_{ij} &= \langle \nabla_i \nu, \varepsilon_j\rangle \\
&= (-1)^\sigma \langle \nabla_i (W/f), \varepsilon_j\rangle \\
&= (-1)^\sigma f^{-2}\langle f\nabla_i W - W\nabla_i f, \varepsilon_j\rangle \\
&= (-1)^\sigma f^{-1}\nabla_i W_j \\
&= -p_{ij}.
\end{aligned}
$$

This shows that $\Sigma$ is a MOTS. The capillarity follows from $\langle \nu, \varepsilon_n\rangle = \kappa$ proven earlier in Lemma 5.5. (Considering asymptotics, $\Sigma \cap \partial M \neq \emptyset$.)

Recall that the hypersurface spinor connection on $\Sigma$ is given by

$$\nabla^\Sigma_i = \nabla_i + \tfrac{1}{2}(\nabla_i \nu)\cdot \nu \cdot .$$

Hence,

$$
\begin{aligned}
&\nabla^\Sigma_i \phi \\
=&\nabla_i \phi + \tfrac{1}{2}(\nabla_i \nu)\cdot \nu \cdot \phi \\
=&-\tfrac{1}{2}(-1)^\sigma p_{ij}\varepsilon_j \cdot \varepsilon_0 \cdot \phi - \tfrac{1}{2}\sum_{j<n} p_{ij}\varepsilon_j \cdot \nu \cdot \phi \\
=&-\tfrac{1}{2}(-1)^\sigma p_{ij}\varepsilon_j \cdot \varepsilon_0 \cdot \phi + \tfrac{1}{2}(-1)^\sigma \sum_{j<n} p_{ij}\varepsilon_j \cdot \varepsilon_0 \cdot \phi \\
=&-\tfrac{1}{2}(-1)^\sigma p_{in}\varepsilon_n \cdot \varepsilon_0 \cdot \phi = -\tfrac{1}{2}p_{in}\phi,
\end{aligned}
$$

where we have used $(-1)^\sigma \nu \cdot \varepsilon_0 \cdot \phi = \phi$ and $\nabla_i \nu = -\sum_j p_{ij}\varepsilon_j$. Therefore,

$$\nabla^\Sigma_i (\phi|\phi|^{-1}) = |\phi|^{-1}\nabla^\Sigma_i \phi - \tfrac{1}{2}|\phi|^{-3}\phi(\langle \nabla^\Sigma_i \phi, \phi\rangle + (\phi, \nabla^\Sigma_i \phi)) = 0.$$

That is, $|\phi|^{-1}\phi$ is a parallel spinor with respect to the induced spinor connection of $\Sigma$. □

Now we determine the equations satisfied by the boundary $\partial M$, in particular, we determine all the components of symmetric 2-tensor $h^{\partial M} + \kappa p_{|\partial M}$.

**Lemma 5.10.** *Let $\{e_i\}$ be an orthonormal frame defined near $\partial M$ such that $e_n$ is the unit outward normal of $\partial M$ in $M$, and $e_1$ be such that $e_1 = (-1)^{\sigma+1}\xi^\top|\xi^\top|^{-1}$, then the only nonzero components of the symmetric 2-tensor $h^{\partial M} + \kappa p_{|\partial M}$ is $h_{11}^{\partial M} + \kappa p_{11} = \tau p_{n1}$ and $p_{ni=0}$ for $i \neq 1, i \neq n$ along $\partial M$.*

*Proof.* Since $Q\phi = (-1)^\sigma\phi$ along $\partial M$ and $\tilde{\nabla}_\alpha\phi = 0$, we see

$$\tilde{\nabla}_\alpha(Q\phi) = 0.$$

Expanding by using the definitions of $Q$ and $\tilde{\nabla}$, we see

$$\begin{aligned}
0 =& \tilde{\nabla}_\alpha(Q\phi) \\
=& (\nabla_\alpha + \tfrac{1}{2}(-1)^\sigma p_{\alpha j}e_j \cdot e_0 \cdot)Q\phi \\
=& (\nabla_\alpha Q)\cdot\phi + Q\cdot\nabla_\alpha\phi + \tfrac{1}{2}(-1)^\sigma p_{\alpha j}e_j\cdot e_0\cdot Q\cdot\phi \\
=& \kappa\nabla_\alpha e_n\cdot e_0\cdot\phi + \tau(-1)^{\chi_1}\sqrt{-1}\nabla_\alpha e_n\cdot\phi \\
& + \tfrac{1}{2}(-1)^\sigma p_{\alpha j}(e_j\cdot e_0\cdot Q\cdot\phi - Q\cdot e_j\cdot e_0\cdot\phi) \\
=& \kappa h_{\alpha\beta}e_\beta\cdot e_0\cdot\phi + \tau(-1)^{\chi_1}h_{\alpha\beta}\sqrt{-1}e_\beta\cdot\phi \\
& + \tfrac{1}{2}(-1)^\sigma p_{\alpha\beta}(e_\beta\cdot e_0\cdot Q\cdot\phi - Q\cdot e_\beta\cdot e_0\cdot\phi) \\
& + \tfrac{1}{2}(-1)^\sigma p_{\alpha n}(e_n\cdot e_0\cdot Q\cdot\phi - Q\cdot e_n\cdot e_0\cdot\phi) \\
=& \kappa h_{\alpha\beta}e_\beta\cdot e_0\cdot\phi + \tau(-1)^{\chi_1}h_{\alpha\beta}\sqrt{-1}e_\beta\cdot\phi \\
& - \kappa p_{\alpha\beta}(-1)^\sigma e_\beta\cdot e_n\cdot\phi \\
& + \tau p_{\alpha n}(-1)^{\sigma+\chi_1}\sqrt{-1}e_0\cdot\phi.
\end{aligned}$$

We calculate $0 = \langle e_\gamma\cdot\tilde{\nabla}_\alpha(Q\phi), \phi\rangle$, and obtain

$$\begin{aligned}
0 =& \tau(-1)^{\sigma_1}\sqrt{-1}h_{\alpha\beta}\langle e_\beta\cdot\phi, e_\gamma\cdot\phi\rangle \\
& - \kappa p_{\alpha\beta}(-1)^\sigma\langle e_\beta\cdot e_n\cdot\phi, e_\gamma\cdot\phi\rangle \\
& + \tau p_{\alpha n}(-1)^{\sigma+\chi_1}\sqrt{-1}\langle e_0\cdot\phi, e_\gamma\cdot\phi\rangle \\
=& -\tau(-1)^{\chi_1}\sqrt{-1}h_{\alpha\beta}\langle e_\gamma\cdot e_\beta\cdot\phi,\phi\rangle \\
& - \kappa p_{\alpha\beta}\tau(-1)^{\chi_1}\sqrt{-1}\langle e_\gamma\cdot e_\beta\cdot\phi,\phi\rangle \\
& - \tau p_{\alpha n}(-1)^{\sigma+\chi_1}\sqrt{-1}\langle e_\gamma\cdot e_0\cdot\phi,\phi\rangle \\
=& -\tau(-1)^{\chi_1}\sqrt{-1}h_{\alpha\beta}\langle e_\gamma\cdot e_\beta\cdot\phi,\phi\rangle \\
& - \kappa p_{\alpha\beta}\tau(-1)^{\chi_1}\sqrt{-1}\langle e_\gamma\cdot e_\beta\cdot\phi,\phi\rangle \\
& - \tau^2 p_{\alpha n}(-1)^{\sigma+\chi_1+\sigma+\chi_1}\sqrt{-1}\langle\sqrt{-1}\cdot e_\gamma\cdot e_0\cdot e_n\cdot\phi,\phi\rangle
\end{aligned}$$

Dividing the above by $\tau(-1)^{\sigma_1}\sqrt{-1}$,

$$(h_{\alpha\beta} + \kappa p_{\alpha\beta})\langle e_\gamma\cdot e_\beta\cdot\phi,\phi\rangle - \tau p_{\alpha n}(-1)^{\chi_1}\langle\sqrt{-1}e_\gamma\cdot e_0\cdot e_n\cdot\phi,\phi\rangle = 0.$$

Taking the real part,

$$-(h_{\alpha\gamma} + \kappa p_{\alpha\gamma}) - \tau p_{\alpha n}(-1)^{\sigma_1}\langle\sqrt{-1}e_\gamma\cdot e_0\cdot e_n\cdot\phi,\phi\rangle = 0.$$

By symmetry,

$$-(h_{\alpha\gamma} + \kappa p_{\alpha\gamma}) - \tau p_{\gamma n}(-1)^{\sigma_1}\langle\sqrt{-1}e_\alpha\cdot e_0\cdot e_n\cdot\phi,\phi\rangle = 0.$$

Taking $\gamma = 1$ and applying (5.7) gives

$$-(h_{\alpha 1} + \kappa p_{\alpha 1}) + \tau p_{\alpha n} = 0.$$

Taking $\gamma \neq 1$ and applying (5.7) leads to

$$h_{\alpha\gamma} + \kappa p_{\alpha\gamma} = 0 \text{ for } \gamma \neq 1.$$

Now since the equality of (1.2) is achieved, we see $p_{\alpha n} = 0$ for $\alpha \neq 1$. □

**Lemma 5.11.** *Let $\Sigma$ be a leaf of the foliation given in Proposition 5.9, then $\partial\Sigma$ is totally geodesic in $\Sigma$.*

*Proof.* Let $\nu = (-1)^\sigma \xi$, then $\tilde{\zeta} = \eta - \langle \eta, \nu\rangle \nu$ is normal to $\partial\Sigma$ in $\Sigma$. And the norm of this vector is

$$|\tilde{\zeta}|^2 = |\eta - \langle \eta, \nu\rangle \nu|^2 = |\eta|^2 - \langle \eta, \nu\rangle^2 = \tau^2$$

which is a constant. Let $2 \leqslant i, j \leqslant n-1$, then the vector field $e_i$ and $e_j$ is parallel to $\partial\Sigma$. We verify the following

$$\begin{aligned}&\langle \nabla_i \tilde{\zeta}, e_j\rangle \\ =&\langle \nabla_i \eta - \langle \eta, \nu\rangle \nabla_i \nu, e_j\rangle \\ =&h_{ij}^{\partial M} - \langle \eta, \nu\rangle h_{ij}^{\Sigma} \\ =&h_{ij}^{\partial M} + \kappa p_{ij} = 0\end{aligned}$$

by Proposition 5.9 and Lemma 5.10. Hence, $\partial\Sigma$ is totally geodesic in $\Sigma$. □

Now we can actually have the following structure for the original initial data set $(M, g, p)$.

**Theorem 5.12.** *There exists a globally defined function $u$ such that $u \to -\kappa x_n - \tau x_1$ as $|x \mapsto \infty$ such that each level set $\Sigma$ is a flat capillary MOTS. Moreover, let $\{\varepsilon_n\}$ be an orthonormal frame such that $\varepsilon_n = \nu$ along $\Sigma$. We set in this theorem that the components of geometric quantities are take with respect to the frame $\{\varepsilon_i\}$. Define*

$$\hat{R}_{ijkl} = R_{ijkl} + p_{jk} p_{il} - p_{ik} p_{jl}.$$

*Then*

$$\begin{aligned}\nabla_i p_{jn} - \nabla_j p_{in} = 0,& \\ \hat{R}_{ijkl} = \nabla_i p_{jk} - \nabla_j p_{ik} &\text{ for all } i, j, k, l \\ \hat{R}_{ijkl} = 0 &\text{ for all } i < n,\ j < n, k,\ l.\end{aligned}$$

*Proof.* Let $m$ be the dimension of the spinor bundle of $\mathcal{S}^{n,1}$, see Section 3.1 (in fact, $m = 2^{[n/2]+1}$). Let $\{\phi_{0,i}\}_{1 \leqslant i \leqslant m}$ be an orthonormal basis of the constant spinors such that

$$\sqrt{-1} \tfrac{\partial}{\partial x^1} \cdot \tfrac{\partial}{\partial x^0} \cdot \tfrac{\partial}{\partial x^n} \cdot \phi_{0,i} = (-1)^{\sigma_{1,i}+1} \phi_{0,i},$$

where $\sigma_{1,i} = -1$ if $1 \leqslant i \leqslant m/2$ and $\sigma_{1,i} = 0$ if $m/2 < i \leqslant m$. We can obtain a set of spacetime spinors $\{\phi_i\}_{1 \leqslant i \leqslant m}$ such that each $\phi_i$ is asymptotic to $\phi_{0,i}$. Each $\phi_i$ defines a foliation, say $\mathcal{F}_i$ of $M$, by Lemma 5.7 and Proposition 5.8, the foliations $\{\mathcal{F}_i\}$ are the same one. By Proposition 5.8, $\{\phi_i |\phi_i|^{-1}\}$ forms an orthonormal basis of parallel spinors along each leaf $\Sigma$. Hence, $\Sigma$ must be flat. Moreover, the boundary $\partial\Sigma$ is totally geodesic. Hence each leaf $\Sigma$ is flat $\mathbb{R}^{n-1}_+$ and $M$ is topologically $\mathbb{R}^n_+$.

By the Poincaré lemma, $(-1)^\sigma W = \nabla u$ for some global function $u$ where $\sigma$ and $W$ are associated with any choice of spinor from the set $\{\phi_i\}$.

With suitable scaling, we see $u \to -\kappa x_n - \tau x_1$ as $|x| \to \infty$ from (5.8). For any $m$-tuple $s = (s_1, \ldots, s_m)$ of spinors and any Euclidean vector $X$, we define a linear transformation $\omega_X$ by

$$(\omega_X s)_i = \sum_{j=1}^{m} (\omega_X)_{ij} s_j = \langle X \cdot \tfrac{\partial}{\partial x^0} \cdot \phi_{0,j}, \phi_{0,i} \rangle s_j.$$

Let $N_0$ be the limit of $\nu$. We set $s = (\phi_1, \ldots, \phi_m)$, then by (5.6) and Lemma 5.7,

$$\omega_{N_0} s = \nu \cdot e_0 \cdot s.$$

Moreover,

$$\nabla_i s + \tfrac{1}{2} p_{ij} e_j \cdot e_0 \cdot \omega_{N_0} s = 0,$$

since the components of $s$ solves (5.2). The formal notation of $\omega_{N_0}$ and Proposition 5.8 then allows us to re-use the calculation in [CW24, Section 3.6] which finishes the proof. □

*Remark* 5.13. It follows from (5.5) that $\Delta u + \mathrm{tr}_g\, p |\nabla u| = 0$. The function $u$ is called a *spacetime harmonic function* introduced by [HKK22].

5.4. **Construction of pp-wave spacetime metric.** We use the spacetime harmonic function $u$ in Theorem 5.12 and find canonical coordinates $\{y^\alpha\}_{1 \leqslant \alpha \leqslant n-1}$ on its level sets.

We set $y^n = u$ and we calculate the metric in terms of $\{y^i\}_{1 \leqslant i \leqslant n}$. For $y^1$, we solve

$$\Delta_\Sigma y^1 = 0 \text{ in } \Sigma,\ y^1 = \tfrac{\kappa}{\tau} y^n \text{ along } \partial\Sigma,\ y^1 = \tau x^n - \kappa x^1 + o(1). \tag{5.9}$$

For $y^i$ with $2 \leqslant i \leqslant n-1$, we solve

$$\Delta_\Sigma y^i = 0 \text{ in } \Sigma,\ \tfrac{\partial y^i}{\partial y^1} = 0 \text{ along } \partial\Sigma,\ y^i = x^i + o(1).$$

Note that by (5.9), $\frac{\partial}{\partial y^1}$ is orthogonal to $\partial\Sigma$ in $\Sigma$.

*Remark* 5.14. We are less concerned with the decay rates of $\{y^i\}$ and later such quantities as $N$, $Y$, $\ell$ and $F$, which were well studied in [HZ24]. The reason, in our case, is that it is enough to take the boundary version of various estimates (see [ABdL16, Section 3]) into consideration as well.

We have another set of asymptotically flat coordinates $\{z^i\}$ on $M$ related to $\{y^i\}$ by

$$z^1 = -\tau y^n - \kappa y^1,\ z^i = y^i \text{ if } 2 \leqslant i \leqslant n-1,\ z^n = -\kappa y^n + \tau y^1. \tag{5.10}$$

See Figure 5.1.

Note that $\{z^i\}$ models on the original asymptotically flat coordinates $\{x^i\}$, and they are asymptotic to each other. We set $N = |\nabla u|^{-1}$ and $Y_\alpha = g\left(\frac{\partial}{\partial y^n}, \frac{\partial}{\partial y^\alpha}\right)$. The metric $g$ is now given by

$$g = (N^2 + |Y|^2)(\mathrm{d}y^n)^2 + 2Y_\alpha \mathrm{d}y^n \mathrm{d}y^\alpha + \sum_{\alpha=1}^{n-1} (\mathrm{d}y^\alpha)^2. \tag{5.11}$$

The inverse metric is then

$$g^{-1} = \begin{bmatrix} N^{-2} & -N^{-2} Y^T \\ -N^{-2} Y & I_{n-1} + N^{-2} Y Y^T \end{bmatrix}. \tag{5.12}$$

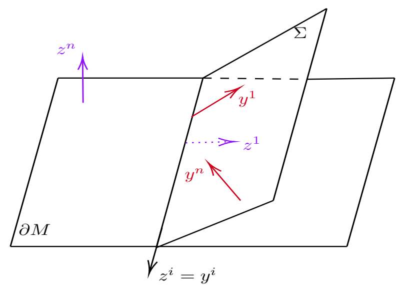


FIGURE 5.1. Relations between $\{y^i\}$ and $\{z^i\}$.

The findings of Hirsch-Zhang [HZ24, Lemmas 5.3-5.5] regarding (5.11) are collected below.

**Lemma 5.15.** *Let $\Sigma$ be any $y^n$-level set, there exists some function $\ell$ such that $Y_\alpha = \nabla^\Sigma_\alpha \ell$. And $(M, g)$ arises as the $\{t = -\ell\}$ spacelike slice of the pp-wave spacetime metric (see Definition 1.6)*

$$\tilde{g} = -2\mathrm{d}t\mathrm{d}y^n + F(y)(\mathrm{d}y^n)^2 + \sum_\alpha (\mathrm{d}y^\alpha)^2, \tag{5.13}$$

*where $y = (y^1, \dots, y^n)$ and $y^1 \geqslant \frac{\kappa}{\tau} y^n$ (since $z^n \geqslant 0$), and $F = N^2 + |\nabla^\Sigma \ell|^2 - 2\ell_u$ is superharmonic on $\Sigma$.*

*Remark* 5.16. The Killing the development is given by $\tilde{g} = 2\mathrm{d}\tau\mathrm{d}u + g$ on $M \times \mathbb{R} = \mathbb{R}^{n+1}_+$ and (5.13) is obtained by setting $\tau = -t - \ell$. And $F$ is superharmonic due to

$$\mu = -\tfrac{1}{2} N^{-2} \Delta_\Sigma F \tag{5.14}$$

and the dominant energy condition (1.1). We also have the helpful relation

$$p_{\alpha\beta} = N^{-1}\ell_{\alpha\beta},\ p = -|\nabla y^n|^{-1}\nabla^2 y^n = -N\nabla^2 y^n. \tag{5.15}$$

Now we derive an equation satisfied by $Y_1$ and $N$ on the boundary. We calculate all the components of the second fundamental form $h^{\partial M} + \kappa p_{|\partial M}$ of $\partial M$ in $M$. To this end, we need the Christoffel symbols of the metric, which is recorded in Lemma A.1.

**Lemma 5.17.** *Let $\ell$ and $N$ be given as above, then*

$$|\nabla z^n| = 1 \text{ and } \kappa + \tau Y_1 = \kappa N, \tag{5.16}$$

*along $\partial M$.*

*Proof.* The vector field $\frac{\partial}{\partial y^1}$ is normal to $\partial \Sigma$ in $\Sigma$ by construction, and $\nabla z^n |\nabla z^n|^{-1}$ is the unit inward normal of $\partial M$ in $M$. Since $\Sigma$ and $\partial M$ form a contact angle $\theta$ by Proposition 5.9, then

$$\left\langle \tfrac{\partial}{\partial y^1}, \tfrac{\nabla z^n}{|\nabla z^n|} \right\rangle = \tau,$$

which gives $|\nabla z^n| = 1$. So using (5.12),

$$\begin{aligned} 1 = |\nabla z^n|^2 = & |-\kappa \nabla y^n + \tau \nabla y^1|^2 \\ = & \kappa^2 g^{nn} - 2\kappa\tau g^{n1} + \tau^2 g^{11} \\ = & \kappa^2 N^{-2} + 2\kappa\tau N^{-2} Y_1 + \tau^2 (1 + N^{-2} Y_1^2). \end{aligned}$$

By solving the above, $\tau Y_1 = -\kappa \pm \kappa N$. Considering the asymptotics, it is only possible that $\kappa + \tau Y_1 = \kappa N$. □

We now express the boundary curvatures $h^{\partial M}$ in terms of the metric.

**Lemma 5.18.** *We have*

$$(h^{\partial M} + \kappa p)(\tfrac{\partial}{\partial z^1}, \tfrac{\partial}{\partial y^\alpha}) = h^{\partial M}_{\alpha\beta} + \kappa p_{\alpha\beta} = 0,$$
$$(h^{\partial M} + \kappa p)(\tfrac{\partial}{\partial z^1}, \tfrac{\partial}{\partial z^1}) = -\tfrac{1}{2}\tau^3 F_1,$$

*along $\partial M$ for $\alpha \neq 1$, $\beta \neq 1$.*

*Proof.* The unit outward normal of $\partial M$ in $M$ is given by $-\nabla z^n$, so

$$\begin{aligned} & h^{\partial M}_{\alpha\beta} \\ = & -\nabla_\alpha \nabla_\beta z^n \\ = & \kappa \nabla_\alpha \nabla_\beta u + \tau \nabla_\alpha \nabla_\beta y^1 \\ = & -\kappa N^{-2} \ell_{\alpha\beta} + \tau \Gamma^1_{\alpha\beta} \\ = & -\kappa N^{-2} \ell_{\alpha\beta} - \tau N^{-2} \ell_1 \ell_{\alpha\beta}, \end{aligned}$$

for $\alpha \neq 1$, $\beta \neq 1$. Since $p_{\alpha\beta} = N^{-1}\ell_{\alpha\beta}$, $h^{\partial M}_{\alpha\beta} + \kappa p_{\alpha\beta} = N^{-2}\ell_{\alpha\beta}(\kappa N - \kappa - \tau \ell_1)$ which vanishes by (5.16).

First, we have that

$$h^{\partial M} + \kappa p = (k - \kappa N)\nabla^2 y^n - \tau \nabla^2 y^1 \tag{5.17}$$

restricted to $T\partial M \otimes T\partial M$ using (5.10) and (5.16). Hence,

$$\begin{aligned} & h^{\partial M}(\tfrac{\partial}{\partial z^1}, \tfrac{\partial}{\partial y^\alpha}) + \kappa p(\tfrac{\partial}{\partial z^1}, \tfrac{\partial}{\partial y^\alpha}) \\ = & (\kappa N - \kappa)(\nabla^2 y^n)(\tau \tfrac{\partial}{\partial y^n} + \kappa \tfrac{\partial}{\partial y^1}, \tfrac{\partial}{\partial y^\alpha}) + \tau(\nabla^2 y^1)(\tau \tfrac{\partial}{\partial y^n} + \kappa \tfrac{\partial}{\partial y^1}, \tfrac{\partial}{\partial y^\alpha}) \\ = & (\kappa N - \kappa)(\tau \Gamma^n_{n\alpha} + \kappa \Gamma^n_{1\alpha}) + \tau(\tau \Gamma^1_{n\alpha} + \kappa \Gamma^1_{1\alpha}) \\ = & (\kappa N - \kappa)(\tau \Gamma^n_{n\alpha} + \kappa \Gamma^n_{1\alpha}) - \tau \ell_1 (\tau \Gamma^n_{n\alpha} + \kappa \Gamma^n_{1\alpha}), \end{aligned}$$

where we have used Lemma A.1 in the last line. Hence

$$h^{\partial M}(\tfrac{\partial}{\partial x^1}, \tfrac{\partial}{\partial y^\alpha}) + \kappa p(\tfrac{\partial}{\partial x^1}, \tfrac{\partial}{\partial y^\alpha}) = (\tau \Gamma^n_{n\alpha} + \kappa \Gamma^n_{1\alpha})(-\tau \ell_1 + \kappa N - \kappa)$$

which again vanishes due to (5.16). It remains to show the last item, and for that we need $(\nabla^2 y^n)(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1})$ and $(\nabla^2 y^1)(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1})$. We now calculate,

$$\begin{aligned} & (\nabla^2 y^n)(\tfrac{\partial}{\partial z^1}, \tfrac{\partial}{\partial z^1}) \\ = & (\nabla^2 y^n)(\tau \tfrac{\partial}{\partial y^n} + \kappa \tfrac{\partial}{\partial y^1}, \tau \tfrac{\partial}{\partial y^n} + \kappa \tfrac{\partial}{\partial y^1}) \\ = & -(\tau^2 \Gamma^n_{nn} + 2\kappa\tau \Gamma^n_{1n} + \kappa^2 \Gamma^n_{11}), \end{aligned}$$

and

$$\begin{aligned} & (\nabla^2 y^1)(\tfrac{\partial}{\partial z^1}, \tfrac{\partial}{\partial z^1}) \\ = & (\nabla^2 y^1)(\tau \tfrac{\partial}{\partial y^n} + \kappa \tfrac{\partial}{\partial y^1}, \tau \tfrac{\partial}{\partial y^n} + \kappa \tfrac{\partial}{\partial y^1}) \\ = & -(\tau^2 \Gamma^1_{nn} + 2\kappa\tau \Gamma^1_{1n} + \kappa^2 \Gamma^1_{11}) \\ = & \tau^2(\ell_1 \Gamma^n_{nn} + \tfrac{1}{2}F_1) + 2\kappa\tau \ell_1 \Gamma^n_{1n} + \kappa^2 \ell_1 \Gamma^n_{11}. \end{aligned}$$

In the above, we have used Lemma A.1. Hence,

$$(h^{\partial M} + \kappa p)(\tfrac{\partial}{\partial z^1}, \tfrac{\partial}{\partial z^1}) = (\kappa N - \kappa - \tau \ell_1)(\tau^2 \Gamma^n_{nn} + 2\kappa\tau \Gamma^n_{1n} + \kappa^2 \Gamma^n_{11}) - \tfrac{1}{2}\tau^3 F_1 = -\tfrac{1}{2}\tau^3 F_1.$$

by (5.17) and then (5.16). □

To complement Lemma 5.18, we also calculate $p(e_n, \cdot)$ restricted to $\partial M$. Recall that $e_n = -\nabla z^n$.

**Lemma 5.19.** *We have*

$$p(\tfrac{\partial}{\partial y^\alpha}, \nabla z^n) = 0,$$
$$p(\tfrac{\partial}{\partial z^1}, -\nabla z^n) = -\tfrac{1}{2}N^{-1}\tau^2 F_1. \tag{5.18}$$

*Proof.* We check $p(\frac{\partial}{\partial y^\alpha}, \nabla z^n) = 0$ first. Because of (5.15), we need to calculate $(\nabla^2 u)(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^n})$ and

$$(\nabla^2 u)(\tfrac{\partial}{\partial y^\alpha}, \tfrac{\partial}{\partial y^\beta}) = -\Gamma^n_{\alpha\beta} = -N^{-2}\ell_{\alpha\beta}.$$

And

$$(\nabla^2 u)(\tfrac{\partial}{\partial y^n}, \tfrac{\partial}{\partial y^\alpha}) = -\Gamma^n_{n\alpha} = -\tfrac{1}{2}N^{-2}(N^2 + |\nabla^\Sigma \ell|^2)_{,\alpha}.$$

So

$$\begin{aligned}
&\nabla_\alpha \nabla_{\nabla u} u\\
=&g^{ni}\nabla_\alpha\nabla_i u = g^{nn}\nabla_\alpha\nabla_n u + g^{n\beta}\nabla_\alpha\nabla_\beta u\\
=&-\tfrac{1}{2}N^{-4}(N^2 + |\nabla^\Sigma \ell|^2)_{,\alpha} + (-N^{-2}\ell_\beta)(-N^{-2}\ell_{\alpha\beta})\\
=&-N_\alpha N^{-3}.
\end{aligned}$$

and

$$\begin{aligned}
&\nabla_\alpha \nabla_{\nabla y^1} u\\
=&g^{1n}\nabla_\alpha\nabla_n u + g^{1\beta}\nabla_\alpha\nabla_\beta u\\
=&(-N^{-2}\ell_1)(-\tfrac{1}{2}N^{-2}(N^2 + |\nabla^\Sigma \ell|^2)_\alpha) + (\delta^{1\beta} + N^{-2}\ell_1\ell_\beta)(-N^{-2}\ell_{\alpha\beta}).
\end{aligned}$$

Then

$$\begin{aligned}
&\nabla_\alpha \nabla_{\nabla z^n} u\\
=&\nabla_\alpha \nabla_{-\kappa\nabla y^n + \tau\nabla y^1} u\\
=&N^{-1}(\kappa N_\alpha N^{-2} + \tau N_\alpha \ell_1 N^{-2} - \tau N^{-1}\ell_{1\alpha})\\
=&-N^{-1}(\kappa N^{-1} + \tau\ell_1 N^{-1})_{,\alpha} = 0
\end{aligned}$$

by (5.16). The calculation of the right hand side of (5.18) is the most involved in this proof. Indeed,

$$\begin{aligned}
&N\tau p(\tfrac{\partial}{\partial z^1}, -\nabla z^n)\\
=&N^2\tau(\nabla^2 u)(\tfrac{\partial}{\partial z^1}, \nabla z^n)\\
=&-N^2\tau(\nabla^2 u)(\tau\tfrac{\partial}{\partial y^n} + \kappa\tfrac{\partial}{\partial y^1}, -\kappa\nabla y^n + \tau\nabla y^1)\\
=&-N^2\tau(-\kappa\tau\Gamma^n_{ni}g^{ni} - \kappa^2\Gamma^n_{1i}g^{ni} + \tau^2\Gamma^n_{ni}g^{1i} + \kappa\tau\Gamma^n_{1i}g^{1i})\\
=&-N^2\tau(-\kappa\tau\Gamma^n_{nn}g^{nn} - \kappa^2\Gamma^n_{1n}g^{nn} + \tau^2\Gamma^n_{nn}g^{1n} + \kappa\tau\Gamma^n_{1n}g^{1n})\\
&\quad - N^2\tau(-\kappa\tau\Gamma^n_{n\alpha}g^{n\alpha} - \kappa^2\Gamma^n_{1\alpha}g^{n\alpha} + \tau^2\Gamma^n_{n\alpha}g^{1\alpha} + \kappa\tau\Gamma^n_{1\alpha}g^{1\alpha}).
\end{aligned}$$

Using the two consequences of (5.16) $-\kappa g^{nn}+\tau g^{1n}=-\kappa N^{-1}$ and $-\kappa g^{n\alpha}+\tau g^{1\alpha}=\tau\delta_{1\alpha}+\kappa N^{-1}\ell_\alpha$,

$$\begin{aligned}
&N\tau p(\tfrac{\partial}{\partial z^1},-\nabla z^n)\\
=&N^2\tau(\kappa\tau N^{-1}\Gamma_{nn}^n+\kappa^2N^{-1}\Gamma_{1n}^n)\\
&-N^2\tau[\tau\Gamma_{n\alpha}^n(\tau\delta_{1\alpha}+\kappa N^{-1}\ell_\alpha)+\kappa\Gamma_{1\alpha}^n(\tau\delta_{1\alpha}+\kappa N^{-1}\ell_\alpha)]\\
=&N\kappa\tau(\tau\Gamma_{nn}^n+\kappa\Gamma_{1n}^n)\\
&-N^2\tau(\tau^2\Gamma_{n1}^n+\kappa\tau N^{-1}\ell_\alpha\Gamma_{n\alpha}^n+\kappa\tau\Gamma_{11}^n+\kappa^2N^{-1}\ell_\alpha\Gamma_{1\alpha}^n)\\
=&N\kappa\tau^2(\Gamma_{nn}^n-\ell_\alpha\Gamma_{n\alpha}^n)+N\kappa^2\tau\Gamma_{1n}^n\\
&-N^2\tau^3\Gamma_{n1}^n-N^2\kappa\tau^2\Gamma_{11}^n-N\kappa^2\tau\ell_\alpha\Gamma_{1\alpha}^n\\
=&\kappa\tau^2N_n+\tfrac{1}{2}\kappa^2\tau N^{-1}(N^2+|\nabla^\Sigma\ell|^2)_{,1}\\
&-\tfrac{1}{2}\tau^3(N^2+|\nabla^\Sigma\ell|^2)_{,1}-\kappa\tau^2\ell_{11}-N^{-1}\kappa^2\tau\ell_\alpha\ell_{1\alpha}\\
=&-\tfrac{1}{2}\tau^3F_1.
\end{aligned}$$

In the last line, we have used

$$0=\tfrac{\partial}{\partial z^1}(\kappa f-\tau\ell_1)=\kappa\tau N_n-\tau^2\ell_{11}+\kappa^2N_1-\tau^2\ell_{1n},$$

which is a consequence of (5.16) and that $\frac{\partial}{\partial z^1}$ is tangential to $\partial M$. □

*Remark* 5.20. The vector field

$$\tilde e_1:=\tfrac{\partial}{\partial z^1}-\sum_{\alpha=2}^{n-1}\left\langle\tfrac{\partial}{\partial z^1},\tfrac{\partial}{\partial y^\alpha}\right\rangle\tfrac{\partial}{\partial y^\alpha}=\tfrac{\partial}{\partial z^1}+\sum_{\alpha=2}^{n-1}\tau\ell_\alpha\tfrac{\partial}{\partial y^\alpha}$$

is normal to $\partial\Sigma$ in $\partial M$ and it has length

$$\begin{aligned}
&|\tilde e_1|^2\\
=&\langle\tfrac{\partial}{\partial z^1},\tfrac{\partial}{\partial z^1}\rangle-\tau^2(|\nabla^\Sigma\ell|^2-\ell_1^2)\\
=&\tau^2g_{nn}+2\kappa\tau g_{1n}+\kappa^2g_{11}-\tau^2(|\nabla^\Sigma\ell|^2-\ell_1^2)\\
=&\tau^2(N^2+|\nabla^\Sigma\ell|^2)+2\kappa\tau\ell_1+\kappa^2-\tau^2(|\nabla^\Sigma\ell|^2-\ell_1^2)\\
=&\tau^2N^2+(\tau^2\ell_1^2+2\kappa\tau\ell_1+\kappa^2)\\
=&\tau^2N^2+(\tau\ell_1+\kappa)^2=\tau^2N^2+\kappa^2N^2=N^2,
\end{aligned}$$

where in the last line, we used (5.16). Hence $|\tilde e_1|=N$. Let $e_1=N^{-1}\tilde e_1$, we see

$$h^{\partial M}(e_1,e_1)+\kappa p(e_1,e_1)=\tau p(e_1,e_n),$$

where $e_n=-\nabla z^n$ from

$$(h^{\partial M}+\kappa p)(\tfrac{\partial}{\partial z^1},\tfrac{\partial}{\partial z^1})=N\tau p(\tfrac{\partial}{\partial z^1},-\nabla z^n),$$

which checks out with Lemma 5.10.

We now provide a proof of Theorem 5.21 which is an expanded version of Theorem 1.7 by summarizing the results proven so far.

**Theorem 5.21.** *Let* $(M,g,p)$ *be a spin asymptotically flat initial data set, which satisfies the dominant energy condition* (1.1)*, the tilted dominant energy condition* (1.2) *and has zero mass, i.e.*

$$E+\cos\theta P_n=\sin\theta|\hat P|,$$

*then $(M, g)$ admits a set of coordinates $\{y^i\}$ such that*

*(a) the metric is of the form*

$$g = (N^2 + |\nabla^\Sigma \ell|^2)(\mathrm{d}y^n)^2 + 2\ell_\alpha \mathrm{d}y^\alpha \mathrm{d}y^n + \sum_\alpha (\mathrm{d}y^\alpha)^2,$$

*and satisfies*

$$\cos\theta + \sin\theta \ell_1 = \cos\theta N \text{ along } \partial M;$$

*(b) the $y^n$-level sets are flat capillary MOTS;*

*(c) $(M, g, p)$ isometrically embeds into*

$$\left(M \times \mathbb{R}, \tilde{g} = -2\mathrm{d}t\mathrm{d}y^n + F(y)(\mathrm{d}y^n)^2 + \sum_\alpha (\mathrm{d}y^\alpha)^2\right)$$

*with $t = -\ell$ over the $\{t = 0\}$-slice in $(M \times \mathbb{R}, \tilde{g})$ with the the second fundamental form given by $p = -|\nabla y^n|^{-1}\nabla^2 y^n$; moreover, $F(y) = N^2 + |\nabla^\Sigma \ell|^2 - 2\frac{\partial \ell}{\partial y^n}$ is superharmonic on any $y^n$-level set and $-\frac{\partial F}{\partial y^1} \geqslant 0$ along $\partial M$;*

*(d) moreover if $E + \cos\theta P_n = 0$, then $(M, g, p)$ lies in the half Minkowski space, more specifically,*

$$\left(M \times \mathbb{R}, \tilde{g} = -2\mathrm{d}t\mathrm{d}y^n + (\mathrm{d}y^n)^2 + \sum_\alpha (\mathrm{d}y^\alpha)^2\right)$$

*where the boundary $\partial(M \times \mathbb{R})$ is given by the relation $y^1 = \frac{\kappa}{\tau} y^n$. Moreover, $h^{\partial M} + \kappa p|_{\partial M}$ vanishes and $p(\eta, \cdot)^\top$ vanishes along $\partial M$.*

*Proof.* First, (a) follows by Theorem 5.15, and (b) is already in Proposition 5.9. The item (c) follows from Lemma 5.15, and $-\frac{\partial F}{\partial y^1} \geqslant 0$ follows from Lemma 5.18 and the tilted boundary dominant energy condition (1.2).

It remains to show (d), we observe that $|\hat{P}| = 0$, so we can make free choices of $\frac{\partial}{\partial x^1}$ in Subsection 5.1. Following once again all the proof, we see from Theorem 5.12 that $\hat{R}_{ijkl} = 0$ for all $i$, $j$, $k$, $l$ and from (5.14) that $\Delta_\Sigma F = 0$; it follows from Lemma 5.18 and the free choices of $\frac{\partial}{\partial x^1}$ that $\frac{\partial F}{\partial y^1} = 0$ along $\partial\Sigma$. Since $F$ asymptotics to 1, by the Liouville theorem, $F \equiv 1$. And this finishes the proof. □

## Appendix A. Calculation of Christoffel symbols

In this appendix, we record the Christoffel symbols of the metric (5.11) where $Y_\alpha = \nabla^\Sigma_\alpha \ell$.

**Lemma A.1.** *Let $g = (N^2 + |\nabla^\Sigma \ell|^2)\mathrm{d}u^2 + 2\ell_\alpha \mathrm{d}u\mathrm{d}y^\alpha + |\mathrm{d}y|^2$. The Christoffel symbols of $g$ satisfies the following relations:*

$$\begin{aligned}
&\Gamma^n_{\alpha n} = \tfrac{1}{2} N^{-2}(N^2 + |\nabla^\Sigma \ell|^2)_{,\alpha},\ \Gamma^\beta_{\alpha n} = -\ell_\beta \Gamma^n_{n\alpha},\\
&\Gamma^n_{\beta\alpha} = N^{-2}\ell_{\beta\alpha},\\
&\Gamma^n_{nn} = N_n/N + \ell_\beta \Gamma^n_{n\beta},\\
&\Gamma^\gamma_{\alpha\beta} = -N^{-2}\ell_\gamma \ell_{\alpha\beta} = -\ell_\gamma \Gamma^n_{\alpha\beta},\\
&\Gamma^\alpha_{nn} = -\ell_\alpha \Gamma^n_{nn} - \tfrac{1}{2} F_\alpha,
\end{aligned}$$

*where $F = N^2 + |\nabla^\Sigma \ell|^2 - 2\ell_u$.*

*Proof.* The proof is just tedious calculation. First, we calculate $\Gamma_{\alpha n}^n$,

$$\begin{aligned}&\Gamma_{n\alpha}^n\\=&\tfrac{1}{2}g^{nn}(g_{\alpha n,n}+g_{nn,\alpha}-g_{\alpha n,n})+\tfrac{1}{2}g^{n\beta}(g_{\alpha\beta,n}+g_{n\beta,\alpha}-g_{n\alpha,\beta})\\=&\tfrac{1}{2}g^{nn}g_{nn,\alpha}=\tfrac{1}{2}N^{-2}(N^2+|Y|^2)_{,\alpha}.\end{aligned}$$

Next $\Gamma_{\alpha\beta}^{\gamma}$,

$$\begin{aligned}&\Gamma_{\alpha\beta}^{\gamma}\\=&\tfrac{1}{2}g^{\gamma n}(g_{\alpha n,\beta}+g_{\beta\alpha,n}-g_{\alpha\beta,n})+\tfrac{1}{2}g^{\gamma\mu}(g_{\alpha\mu,\beta}+g_{\beta\mu,\alpha}-g_{\alpha\beta,\mu})\\=&\tfrac{1}{2}g^{\gamma n}(g_{\alpha n,\beta}+g_{\beta\alpha,n})=-N^{-2}\ell_\gamma\ell_{\alpha\beta}.\end{aligned}$$

And $\Gamma_{\alpha\beta}^n$,

$$\begin{aligned}&\Gamma_{\alpha\beta}^{n}\\=&\tfrac{1}{2}g^{nn}(g_{\alpha n,\beta}+g_{\beta n,\alpha}-g_{\alpha\beta,n})+\tfrac{1}{2}g^{n\gamma}(g_{\alpha\gamma,\beta}+g_{\beta\gamma,\alpha}-g_{\alpha\beta,\gamma})\\=&\tfrac{1}{2}g^{nn}(g_{\alpha n,\beta}+g_{\beta n,\alpha})=N^{-2}\ell_{\alpha\beta}.\end{aligned}$$

And $\Gamma_{n\alpha}^{\beta}$ and $\Gamma_{nn}^n$ are related to $\Gamma_{\alpha n}^n$ since,

$$\begin{aligned}\Gamma_{n\alpha}^{\beta}&=\tfrac{1}{2}g^{\beta n}(g_{nn,\alpha}+g_{\alpha n,n}-g_{n\alpha,n})+\tfrac{1}{2}g^{\beta\gamma}(g_{\alpha\gamma,n}+g_{n\gamma,\alpha}-g_{n\alpha,\gamma})\\&=\tfrac{1}{2}g^{\beta n}g_{nn,\alpha}=-\tfrac{1}{2}N^{-2}\ell_\beta(N^2+|\nabla^\Sigma\ell|^2)_{,\alpha}=-\ell_\beta\Gamma_{\alpha n}^n.\end{aligned}$$

and

$$\begin{aligned}&\Gamma_{nn}^{n}\\=&\tfrac{1}{2}g^{nn}g_{nn,n}+\tfrac{1}{2}g^{n\alpha}(2g_{n\alpha,n}-g_{nn,\alpha})\\=&\tfrac{1}{2}N^{-2}(N^2+|\nabla^\Sigma\ell|^2)_{,n}-\tfrac{1}{2}N^{-2}\ell_\alpha(2\ell_{\alpha n}-(N^2+|\nabla^\Sigma\ell|^2)_{,\alpha})\\=&N_n/N+\tfrac{1}{2}N^{-2}\ell_\alpha(N^2+|\nabla^\Sigma\ell|^2)_{,\alpha}\\=&N_n/N+\ell_\alpha\Gamma_{\alpha n}^n.\end{aligned}$$

Finally, we verify the relation $\Gamma_{nn}^{\alpha}=-\ell_\alpha\Gamma_{nn}^n-\frac{1}{2}F_\alpha$ in the following:

$$\begin{aligned}&\Gamma_{nn}^{\alpha}\\=&\tfrac{1}{2}g^{\alpha n}g_{nn,n}+\tfrac{1}{2}g^{\alpha\gamma}(2g_{n\gamma,n}-g_{nn,\gamma})\\=&-\tfrac{1}{2}N^{-2}\ell_\alpha(N^2+|\nabla^\Sigma\ell|^2)_{,n}+\tfrac{1}{2}(\delta_{\alpha\gamma}+N^{-2}\ell_\alpha\ell_\gamma)(2\ell_{\gamma n}-(N^2+|\nabla^\Sigma\ell|^2)_{,\gamma})\\=&-\tfrac{1}{2}N^{-2}\ell_\alpha(N^2+|\nabla^\Sigma\ell|^2)_{,n}+\ell_{\alpha n}-\tfrac{1}{2}(N^2+|\nabla^\Sigma\ell|^2)_{,\alpha}\\&\quad+\tfrac{1}{2}N^{-2}\ell_\alpha\ell_\gamma\ell_{\gamma n}-\tfrac{1}{2}N^{-2}\ell_\alpha\ell_\gamma(N^2+|\nabla^\Sigma\ell|^2)_{,\gamma}\\=&-\ell_\alpha\Gamma_{nn}^n-\tfrac{1}{2}F_{,\alpha}.\end{aligned}$$

This finishes the proof. □

## References

[ABdL16] Sérgio Almaraz, Ezequiel Barbosa, and Levi Lopes de Lima. A positive mass theorem for asymptotically flat manifolds with a non-compact boundary. *Communications in Analysis and Geometry*, 24(4):673–715, 2016.

[ACG08] Lars Andersson, Mingliang Cai, and Gregory J. Galloway. Rigidity and Positivity of Mass for Asymptotically Hyperbolic Manifolds. *Annales Henri Poincaré*, 9(1):1–33, 2008.

[AdL20] Sérgio Almaraz and Levi Lopes de Lima. The mass of an asymptotically hyperbolic manifold with a non-compact boundary. *Ann. Henri Poincaré*, 21(11):3727–3756, 2020.

[AdL22] Sergio Almaraz and Levi Lopes de Lima. Rigidity of non-compact static domains in hyperbolic space via positive mass theorems. 2022. ArXiv:2206.09768 [gr-qc].

[AdLM19] Sergio Almaraz, Levi Lopes de Lima, and Luciano Mari. Spacetime positive mass theorems for initial data sets with noncompact boundary. 2019.

[ALY20] Aghil Alaee, Martin Lesourd, and Shing-Tung Yau. Stable Surfaces and Free Boundary Marginally Outer Trapped Surfaces. *ArXiv:2009.07933 [gr-qc, physics:math-ph]*, 2020.

[BC96] Robert Beig and Piotr T. Chruściel. Killing vectors in asymptotically flat space–times. I. Asymptotically translational Killing vectors and the rigid positive energy theorem. *Journal of Mathematical Physics*, 37(4):1939–1961, apr 1996.

[CH03] Piotr T. Chruściel and Marc Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific journal of mathematics*, 212(2):231–264, 2003.

[Cha18] Xiaoxiang Chai. Positive mass theorem and free boundary minimal surfaces. *ArXiv:1811.06254 [math]*, 2018.

[Cha21] Xiaoxiang Chai. Asymptotically hyperbolic manifold with a horospherical boundary. *ArXiv:2102.08889 [gr-qc]*, 2021.

[CW24] Xiaoxiang Chai and Xueyuan Wan. Initial data set rigidity results for polyhedra. *ArXiv:2408.13801*, 2024.

[EHLS16] Michael Eichmair, Lan-Hsuan Huang, Dan A. Lee, and Richard Schoen. The spacetime positive mass theorem in dimensions less than eight. *J. Eur. Math. Soc. (JEMS)*, 18(1):83–121, 2016.

[HH96] S. W. Hawking and Gary T. Horowitz. The gravitational Hamiltonian, action, entropy and surface terms. *Classical Quantum Gravity*, 13(6):1487–1498, 1996.

[HKK22] Sven Hirsch, Demetre Kazaras, and Marcus Khuri. Spacetime harmonic functions and the mass of 3-dimensional asymptotically flat initial data for the Einstein equations. *J. Differential Geom.*, 122(2):223–258, 2022.

[HL20] Lan-Hsuan Huang and Dan A. Lee. Equality in the Spacetime Positive Mass Theorem. *Communications in Mathematical Physics*, 376(3):2379–2407, 2020.

[HL23] Lan-Hsuan Huang and Dan A. Lee. Equality in the spacetime positive mass theorem II. *ArXiv:2302.06040 [gr-qc]*, 2023.

[HL24] Lan-Hsuan Huang and Dan A. Lee. Bartnik mass minimizing initial data sets and improvability of the dominant energy scalar. *Journal of Differential Geometry*, 126(2):741–800, 2024.

[HZ03] Oussama Hijazi and Xiao Zhang. The Dirac-Witten operator on spacelike hypersurfaces. *Comm. Anal. Geom.*, 11(4):737–750, 2003.

[HZ24] Sven Hirsch and Yiyue Zhang. Initial data sets with vanishing mass are contained in pp-wave spacetimes. *ArXiv: 2403.15984 [gr-qc]*, 2024.

[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[Mic11] B. Michel. Geometric invariance of mass-like asymptotic invariants. *Journal of Mathematical Physics*, 52(5):52504, 2011.

[PT82] Thomas Parker and Clifford Henry Taubes. On Witten's proof of the positive energy theorem. *Communications in Mathematical Physics*, 84(2):223–238, 1982.

[Sak21] Anna Sakovich. The Jang Equation and the Positive Mass Theorem in the Asymptotically Hyperbolic Setting. *Communications in Mathematical Physics*, apr 2021.

[SY79] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[Wan01] Xiaodong Wang. The mass of asymptotically hyperbolic manifolds. *Journal of Differential Geometry*, 57(2):273–299, 2001.

[Wit81] Edward Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.



Department of Mathematics, POSTECH, Pohang, Gyeongbuk, South Korea
*Email address*: xxchai@kias.re.kr, xxchai@postech.ac.kr